Week 5: Partition of unity

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1. Tietze extension theorem

The proof of this theorem is very constructive and really a hidden gem in general topology.

1.1. Definition

A topological space X is normal if for $H \subset V \subset X$ with H closed and V open, there is an open U and closed Z with $H \subset U \subset Z \subset V \subset X$.

It's immediate to see if $Y \subset X$ is an open subspace with X normal, then Y is also normal.

We firstly prove an important lemma on compact Hausdorff spaces(note this is not true for locally Hausdorff spaces, as they are not normal!)

1.2. Urysohn's lemma

Let X be a normal space, A_1 and A_2 be disjoint closed subsets in X. Then there exists a continuous function $f : X \to [0, 1]$ with $f|_{A_1} = 0$ and $f|_{A_2} = 1$.

1.2.1. Proof

Consider the chain $A_1 \subset X - A_2 \subset A_2$. Let $r := \frac{s_1}{2^{s^2}} < 1$ with s_1, s_2 positive integers. Inductively by normality of X we construct a chain of sets $U(r)$ and $Z(r)$ such that

$$A \subset U(r) \subset Z(r) \subset X - A_2$$
$$Z(r) \subset U(r'), r < r'$$

This goes in detail as: suppose we have constructed all $U(\frac{s_1}{2^{s_2}})$ and $Z(\frac{s_1}{2^{s_2}})$ for $1 \leq s_1 \leq 2^{s_2 - 1}$, then we may construct

$$Z\left(\frac{s_1}{2^{s_2}}\right) \subset U\left(\frac{2s_1 + 1}{2^{s_2+1}}\right) \subset Z\left(\frac{2s_1 + 1}{2^{s_2+1}}\right)$$
$$U\left(\frac{s_1}{2^{s_2+1}}\right) \subset Z\left(\frac{s_1}{2^{s_2+1}}\right) \subset U\left(\frac{s_1}{2^{s_2}}\right)$$

We define $f(x) = 1$ for $x \notin U(r)$ for any r, otherwise $f(x) = \inf\{r : x \in U(r)\}$. By construction it's clear $f(A_1) = 0$ and $f(A_2) = 1$. It remains to show f is continuous. For this we make two observations:
1. If $y \notin U(s)$, then $y \notin U(r)$ for $s < r$, thus $f(y) = \inf\{r : y \in U(r)\} \geq s$.
2. If $y \in Z(s)$, then $y \in U(r)$ for $r > s$, thus $f(y) = \inf\{r : x \in U(r)\} \leq s$.

Now by the classification of opens in $\mathbb{R}$ we may pick $(p, q) \subset [0, 1]$ and $x \in f^{-1}(p, q)$. By construction we find $r_1 = \frac{a_1}{2^{b_1}}$ and $r_2 = \frac{a_2}{2^{b_2}}$ such that $p < r_1 < f(x) < r_2 < q$. We thus know $x \in U(r_2) - Z(r_1) =: V$. We want to show this is in $f^{-1}(p, q)$, hence it is open. Now as $U(r_2) \subset Z(r_2)$, we have $f(V) \leq r_2 < q$ and as $Z(r_1) \subset U(r_2)$ we have $f(V) \geq r_1 > p$. This finishes the proof. ■

1.3. Tietze extension

Let X be a normal space and $Z \subset X$ closed, assume $f : Z \to \mathbb{R}^n$ continuous, then there exists a function $g : X \to \mathbb{R}^n$ such that $g|_Z = f$.

1.3.1. Proof

WLOG we assume $n = 1$. Let $c_0 := \sup\{|f(x)| : x \in Z\}$, $E_0 := \left\{x \in Z : f(x) \geq \frac{c_0}{3}\right\}$ and $F_0 := \left\{x \in Z : f(x) \leq -\frac{c_0}{3}\right\}$. As f continuous, both E_0 and F_0 are closed. By Urysohn's lemme(and scaling) we construct $f_0 X \to \mathbb{R}$ such that $f_0(E_0) = \frac{c_0}{3}, f_0(F_0) = -\frac{c_0}{3}$ and $f_0(X) \subset \left[-\frac{c_0}{3}, \frac{c_0}{3}\right]$.

Inductively we construct a sequence of functions f_n such that $|f_n(x)| \leq \frac{2^n}{3^{n+1}} c_0$ thus $|f(x) - \sum_{i=0}^{n} f_i(x)| \leq \frac{2^{n+1}}{3^{n+1}} c_0, x \in Z$. This is possible as we can define c_n, E_n and F_n like before for function $f - \sum_{i=0}^{n-1} f_i$.

Now consider $g_n := \sum_{i=0}^{n} f_i$ the partial sum, we have

$$|g_n - g_m| = |\sum_{i=m}^{n} f_i| \leq \sum_{i=m}^{n} \frac{2^i}{3^{i+1}} c_0 \leq \frac{2^m}{3^m} c_0$$

Hence this sequence is a Cauchy sequence and as $\mathbb{R}$ complete, it converges uniformly to $g : X \to \mathbb{R}$ continuous. By construction $|f - g_n|$ converges to 0 on Z, thus $g|_Z = f$. ■

2. Existence of partition of unity

The partition of unity exists for any paracompact Hausdorff space. We will prove this substantially.

2.1. Lemma

Let X be a topological space and $\{U_i\}_{i \in I}$ be an open cover. Suppose we have a locally finite refinement $\left\{V_j\right\}_{j \in J}$ with $\varphi : J \to I$, then $\{W_i\}_{i \in X}$ defined as

$$W_i := \bigcup_{j \in \varphi^{-1}(i)} V_j$$

is still a locally finite refinement of $\{U_i\}$.

2.1.1. Proof

By construction W_i is indeed in U_i, hence a refinement. We need to show it's locally finite. Pick $x \in X$, by assumption $\left\{V_j\right\}$ is locally finite and we have an open neighbourhood U_x for x such that there exists $K \subset J$ finite and $\forall j \notin J, U_x \cap V_j = \emptyset$. Thus if $i \notin \varphi(K)$, then $U_x \cap W_i = \emptyset$, and as $\varphi(K)$ finite, this concludes the proof. ■

2.2. Shrinking Lemma

Let X be a normal topological space and $\{U_i\}$ a locally finite open cover, then there exists an open cover $\{V_i\}$ such that $\overline{V_i} \subset U_i$.

2.2.1. Proof

We want to apply Zorn's lemma. Let S be the set of collections $\mathcal{W}_J = \{W_{i,J}\}$ of open sets such that for $J \subset I$ we have $\overline{W_{i,J}} \subset U_i, i \in J$ and $W_{i,J} = U_i, i \in I - J$, and $\mathcal{W}_J$ is an open cover of X. S is nonempty since X is normal, we can pick any index in I to perform a refinement. Let's define a partial order $\leq$ on S as $\mathcal{W}_J \leq \mathcal{W}_K$ if $J \subset K$ and $W_{i,J} = W_{i,K}$ for each $i \in J$.

Let T be a totally ordered subset of S. Define an index set $K = \bigcup\{J : \mathcal{W}_J \in T\}$ and the collection $\mathcal{W}_K$ as $\forall i \in K, W_{i,K} = W_{i,J}$ if $i \in J$. Since T is totally ordered, this is well-defined and $\overline{W_{i,K}} \subset U_i, i \in K$. Moreover, $\mathcal{W}_K$ is an open cover of X. Since $\{U_i\}$ is locally finite, $\mathcal{W}_K$ is locally finite and at each point $x \in X$ there exists $\{i_1, ..., i_k\}$ with $\{i_1, ..., i_k\} \subset K$ and $x \in \bigcup_{i \in K} W_{i,K}$.

By Zorn's lemma S has a maximal element $\mathcal{V} = \{V_i\}$. The indexing set of $\mathcal{V}$ must be I otherwise we can always enlarge the indexing set using the normality of X. Hence by construction this is our searching open cover. ■

2.3. Existence

By paracompactness of X, each open cover $\{U_i\}$ has a locally finite refinement. By previous lemma we may assume this refinement has the same index. Now it suffices to show every locally finite open cover has a partition of unity.

As paracompact Hausdorff space is normal, we apply the shrinking lemma twice to get two open covers $\{V_i\}$ and $\{W_i\}$ such that

$$W_i \subset \overline{W_i} \subset V_i \subset \overline{V_i} \subset U_i$$

This allows us to apply Urysohn's lemma on two disjoint closed subsets $\overline{W_i}$ and $X - V_i$ to construct a continuous function $h_i : X \to [0, 1]$ such that $h_i\left(\overline{W_i}\right) = 1$ and $h_i(X - V_i) = 0$, hence $\operatorname{supp}(h_i) \subset \overline{V_i} \subset U_i$.

It remains to normalize these functions. We set $h := \sum_{i \in I} h_i$ a finite sum at each point due to the local finiteness of the cover. Moreover $\forall x \in X, h(x) \neq 0$ since $\{W_i\}$ is an open cover of X and $h_i(W_i) = 1$.

Now set $f_i := \frac{h_i}{h}$ and $\sum_{i \in I} f_i = 1$. Thus $\{f_i\}$ is a partition of unity. ■

2.4. Local finiteness

Locally finiteness is actually not crucial in the proof, as suggested by the following lemma:

2.4.1. Theorem(Mather)

Let $\{f_i\}_{i \in I}$ be a partition of unity subordinated to $\{U_i\}_{i \in I}$, then there is a locally finite partition of unity $\{g_i\}_{i \in I}$ subordinated to $\{V_i\}_{i \in I}$ such that $\{V_i\}$ is a refinement of $\{U_i\}$.