Week 4: Metrizable manifolds and distributions

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1. Distributions and foliations

A distribution D is a subbundle of the tangent bundle $\mathrm{T}M$ for a manifold M.

1.1. Properties of a distribution

A distribution D is said to be involutive iff for any section $X, Y \in \Gamma(D)$ we have $[X, Y] \in \Gamma(D)$, where $[.,.]$ is on the total tangent bundle defined.

This condition is purely algebraic, but a very cool theorem gives it a geometric interpretation:

1.1.1. Theorem(Frobenius)

A distribution D is involutive iff it's integrable iff it's complete integrable

1.1.1.1. Proof

s. [[Lee12], Theorem 19.12].

One direction from involution to complete integrable was already shown in the exercise. However we still need to give the precise definition of integrability.

1.1.2. Definition

A immersed manifold $N \neq \emptyset$ in M is called a integral manifold of distribution D if $T_p N = D_p, \forall p \in N$.

1.1.3. Definition

A distribution D is called integrable if $\forall p \in M$, there is a integral manifold N of D with $p \in N$.

1.1.4. Remark

Not every distribution is integrable. For a counterexample, consider the distribution in $\mathbb{R}^3$ linearly spanned by

$$X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, Y = \frac{\partial}{\partial y}$$

It cannot have a integral manifold at the origin because its twisting nature.

People may think the integral manifold as finding an inverse operation of differential(which is a tangential space) on a manifold.

1.1.5. Definition

A distribution D of rank k is completely integrable if there exists a chart (U, φ) for every $p \in M$ such that $\varphi(U)$ is a cube in $\mathbb{R}^n$, and after $D\varphi$, D is generated only by first k standard basis $\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_k}$.

In other word, Lie bracket measures the failure of the integral of D being a euclidean space. As we have a canonical commutative family of sections on $\mathbb{R}^n$ which is $\left\{\frac{\partial}{\partial x_i}\right\}$. If D is involutive, i.e. Lie bracket is closed in D, then by Frobenius D is complete integrable, thus we can always make a chart to "convert" D into $\mathbb{R}^n$.

Distribution has also a deep connection with phase spaces in physics. The bridge between them is so called simplectic geometry. A detailed introduction is beyond the scope. The reader can find all details in [HZ12].

1.2. Foliations

1.2.1. Definition

Let $\mathcal{J}$ be an indexed family, we say $\{N_i\}_{i\in\mathcal{J}}$ is a k-foliation of the manifold M^n if each N_i is a nonempty connected immersed k-submanifold of M such that

1. $N_i \cap N_j = \emptyset, i \neq j$
2. $\bigcup_{i\in\mathcal{J}} N_i = M$

1.2.2. Theorem(Global Frobenius)

Let D be an involutive distribution of M. Then the collection of maximally connected integral manifolds of D forms a foliation of M

It's in general not so easy to produce a foliation. For that, one needs to think about distributions as the kernel of 1-forms and uses contact geometry techniques. Those who have interests may search for Reeb foliations.

2. Manifolds are metrizable

2.1. Smooth case

2.1.1. Theorem(Whitney Embedding)

Let M be a smooth manifold of dimension n, then it can be smoothly embedded into $\mathbb{R}^{2n+1}$.

2.1.1.1. Remark

The best case is in general $\mathbb{R}^{2n}$, of which we need to use foliation to prove. For example, $\mathbb{RP}^2$ cannot be embedded into $\mathbb{R}^3$.

2.1.1.2. Proof

Hard theorem, but for compact manifolds we can prove it relatively easily.

We only show M can be embedded into a sufficiently large euclidean space $\mathbb{R}^N$.

Suppose $M = U_1 \cup \ldots \cup U_k$ be an open cover of charts (U_i, φ_i). As M locally compact, we may choose a finite open cover $\{V_j\}$ of M such that $\overline{V_j} \subset U_j$. Now let $\lambda_j : M \to \mathbb{R}$ be a smooth bump function on V_j, i.e. $\lambda_j|_{V_j} = 1$ and $\operatorname{supp}(\lambda_j) \subset U_j$. We define

$$\psi_j : M \to \mathbb{R}^n, p \mapsto \begin{cases} \lambda_j(p)\varphi_j(p) & \text{if } p \in U_j \\ 0 & \text{else} \end{cases}$$

and set $\theta = \prod_{i=1}^{k} \psi_i \times \prod_{i=1}^{k} \lambda_i : M \to \mathbb{R}^{kn+k}$. We show θ is an embedding.

It's an immersion: suppose $\theta_{*,p}(v) = 0$, then there's a j with $p \in V_j$, thus $\psi_j(p) = \varphi_j(p)$ and $\varphi_{j,*,p}(v) = 0$, but $\varphi_{j,*}$ is an isomorphism, so $v = 0$.

It's injective: Suppose $\theta(p) = \theta(q)$, then there's a j with $p \in V_j$ and $\lambda_j(p) = \lambda_j(q)$, thus $\varphi_j(p) = \varphi_j(q)$, but φ_j is a chart, hence $p = q$.

A complete proof can be found in [[Lee12], Theorem 6.15].

2.1.2. Corollary

Every smooth manifold is metrizable.

2.1.2.1. Proof

By choosing an embedding $i : M \hookrightarrow \mathbb{R}^{2n+1}$, we can get a metric on $i(M)$ induced by euclidean metric. We define the metric on M by setting $d(a, b) = d(i(a), i(b)), a, b \in M$, since i injective, this is a metric.

Alternatively we can firstly define a Riemannian metric $\langle ., . \rangle$ on M as mentioned in week 3. The existence of a Riemannian metric is a direct consequence of partition of ones. We define the distance between $a, b \in M$ to be

$$d(a, b) := \inf_{\gamma : a \to b} l(\gamma)$$

where γ is a piecewise smooth curve in M connecting a and b, and the length is defined to be

$$l(\gamma) = \int_a^b \sqrt{\langle \gamma'(s), \gamma'(s) \rangle} ds$$

Since the length is non-negative, this infimum exists. Man verifies this is indeed a metric, fulfilling the triangle inequality.

2.2. General case

It turns out that every topological manifold is metrizable. The proof is a direct consequence of Urysohn's metrization theorem, whose idea is also finding an embedding into $\mathbb{R}^N$.

2.2.1. Theorem(Urysohn)

Every second-countable regular Hausdorff space is metrizable.

For a proof, see this nLab page: https://ncatlab.org/nlab/show/Urysohn+metrization+theorem.

2.2.2. Easy topological fact

Locally compact Hausdorff spaces are regular.

References

Lee12. Lee, J.M.: Introduction to Smooth Manifolds. Springer New York, NY, New York, USA (2012)

HZ12. Hofer, H., Zehnder, E.: Symplectic Invariants and Hamiltonian Dynamics. Birkhäuser Basel, Basel, CH (2012)