Week 1 : Topological and smooth manifolds

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1 More examples of manifolds

1.1 Lens space

It's better to review some previous knowledge of group actions, also see Section 3.

Let $\mathbb{S}^3 \subset \mathbb{R}^4 \cong \mathbb{C}^2$ be the 3-sphere. We denote the two complex coordinates by (u, v). Let $p, q \in \mathbb{Z}$ with $\gcd(p, q) = 1$. We have a smooth endomorphism of $\mathbb{S}^3$ by

$$T : \mathbb{S}^3 \to \mathbb{S}^3, (u, v) \mapsto \left(e^{\frac{2\pi i}{p}} u, e^{\frac{2\pi i q}{p}} v\right)$$

It's easy to see that $T^p = \mathrm{id}_X$.

We consider the action of $\mathbb{Z}/p$ on $\mathbb{S}^3$ defined by

$$\iota : \mathbb{Z}/p \to C^\infty(X, X), 1 \mapsto T$$

Since $\mathbb{Z}/p$ is cyclic, we know this action is well defined.

By Section 3.3, the orbit space $L(p, q) := \mathbb{S}^3/(\mathbb{Z}/p)$ is a smooth manifold. We call this family the **lens spaces**.

This definition is easy to be generalized into higher dimensions, but in dimension 3 they are already very interesting.

Later we will know the homotopy group of a topological space, which is a powerful algebraic approach to determine whether two topological manifolds are homeomorphic. For dimension $n \leq 2$ we have the following theorem:

1.1.1 Theorem

If X^n and Y^n are homotopic, i.e. all homotopy groups are the same, then they are already homeomorphic.

This is not true in higher dimensions. In 1935, Reidemeister had shown in [Rei35] that $L(5, 1)$ and $L(5, 2)$ are homotopic, but they are not homeomorphic!

1.2 Orientation and projective spaces

The constructions of the torus and the Klein bottle are very similar, yet they yield totally different manifolds, as T^2 can be embedded into $\mathbb{R}^3$ but K^2 not. The essential point here is the orientability.

1.2.1 Definition

Let M^n be a smooth manifold with atlas $\{\varphi_i\}_{i \in I}$. We say M is **orientable** if all chart transition maps $\varphi_i \circ \varphi_j^{-1}$ have positive Jacobian determinant, i.e. $\det\left(\nabla\left(\varphi_i \circ \varphi_j^{-1}\right)\right) > 0, \forall i, j \in I$.

This is rather a naive and computation tendentious definition, we will see more easy definitions later using tangential spaces. Nevertheless, we can still use this to illustrate an interesting example.

1.2.2 Definition

Let K be an $\mathbb{R}$-division algebra, then we consider the quotient $K^n - \{0\}/\sim$ via $x \sim y \Leftrightarrow \exists \lambda \neq 0 \in K, x = \lambda y$. We get the so called n-dimensional K-**projective space** $\mathbb{KP}^n$.

If you have already seen the exercise 4 on sheet 2, you will immediately notice that there's another definition for projective spaces, which can be identified with all 1-dimensional subspaces of K^n, i.e. $\mathrm{Gr}_K(n, 1)$.

1.2.3 Example

Let $K = \mathbb{R}$, we have then the real projective space $\mathbb{RP}^n$. It is orientable iff n is odd. We give a calculation to show $\mathbb{RP}^2$ is not orientable.

Consider the charts $\varphi : \{[x] : x_0 \neq 0\} \to \mathbb{R}^2, [x_0 : x_1 : x_2] \mapsto \left(\frac{x_1}{x_0}, \frac{x_2}{x_0}\right)$ and $\psi : \{[x] : x_1 \neq 0\} \to \mathbb{R}^2, [x_0 : x_1 : x_2] \mapsto \left(\frac{x_0}{x_1}, \frac{x_2}{x_1}\right)$. From the lecture we have $\psi^{-1} : \mathbb{R}^2 \to \{[x] : x_1 \neq 0\}, (x_0, x_2) \mapsto [x_0 : 1 : x_2]$. We calculate the Jacobian of $\varphi \circ \psi^{-1}$:

$$\nabla(\varphi \circ \psi^{-1}) = \begin{pmatrix} \frac{-1}{x^2} & 0 \\ \frac{-y}{x^2} & \frac{1}{x} \end{pmatrix}$$

The determinate is thus $\frac{-1}{x^3}$, which is not always positive.

1.2.4 Example

Let $K = \mathbb{C}$, by a long calculation you can however show that $\mathbb{CP}^n$ is orientable. In fact, more advanced techniques imply that all complex manifolds are orientable with the canonical orientation induced from the complex structure.

2 The Rank Theorem

This section follows Lee's book [Lee12].

2.1 Theorem

Suppose M^m and N^n are smooth manifolds. $f : M \to N$ a smooth map such that the rank of differential of f is constant r everywhere. Then for each $p \in M$ there are charts (U, φ) near p and (V, ψ) near $f(p)$, in this local coordinate f has the from

$$f(x_1, x_2, ..., x_m) = (x_1, ..., x_r, 0, ..., 0)$$

In particular if f is a submersion, then locally it looks like a projection.

2.1.1 Proof

As this is a local theorem, we may assume $M = U \subset \mathbb{R}^m$ and $N = V \subset \mathbb{R}^n$. Since f has constant rank r, we can achieve via reordering that the upper left $r \times r$ submatrix of ∇f is invertible. We rename the coordinate of U as $(x_1, ..., x_r, y_1, ..., y_{m-r})$ and of V as $(v_1, ..., v_r, w_1, ..., w_{n-r})$. The function $f =$

$(Q(x,y), R(x,y))$ where $Q : U \to \mathbb{R}^r$ and $R : U \to \mathbb{R}^{n-r}$. By suitable translation we assume $p = (0,0)$ and $f(p) = (0,0)$.

Define $\varphi : U \to \mathbb{R}^m$ by $\varphi(x,y) = (Q(x,y), y)$. The differential of φ will be

$$\nabla\varphi = \begin{pmatrix} \partial_x Q & \partial_y Q \\ 0 & \mathbb{1}_{m-r} \end{pmatrix}$$

It is invertible at p, thus by inverse function theorem we have $\varphi : U_0 \to U_0'$ a diffeomorphism, where U_0 and U_0' are both neighbourhoods of $(0,0)$. The inverse map is $\varphi^{-1} = (A(x,y), B(x,y))$ for some smooth functions $A : U_0' \to \mathbb{R}^r$ and $B : U_0' \to \mathbb{R}^{m-r}$. We have

$$(x,y) = \varphi(A(x,y), B(x,y)) = (Q(A(x,y), B(x,y)), B(x,y))$$

Hence $B(x,y) = y$ and $\varphi^{-1} = (A(x,y), y)$. On the other hand $Q(A(x,y), y) = x$, and therefore $f \circ \varphi^{-1}$ has the from

$$f \circ \varphi^{-1}(x,y) = (x, R(A(x,y), y)) =: \left(x, \tilde{R}(x,y)\right)$$

The differential of $f \circ \varphi^{-1}$ is

$$\nabla(f \circ \varphi^{-1}) = \begin{pmatrix} \mathbb{1}_r & 0 \\ \partial_x \tilde{R} & \partial_y \tilde{R} \end{pmatrix}$$

Since composing with diffeomorphism does not change the rank, it has rank r. And since the first r columns are linearly independent, we know $\partial_y \tilde{R}$ has to be 0 on U_0', hence $\tilde{R}$ does not depend on y and we let $S(x) = \tilde{R}(x, 0)$.

Finally we need to reduce the V to define a chart. Let $V_0 = \{(v,w) \in V : (v,0) \in U_0'\}$, we thus have $f \circ \varphi^{-1}(U_0') \subset V_0$ and $f(U_0) \subset V_0$. Let $\psi : V_0 \to \mathbb{R}^n$ to be $\psi(v,w) = (v, w - S(v))$. The inverse is given by $\psi^{-1}(s,t) = (s, t + S(s))$. We have shown that (V_0, ψ) is a chart and

$$\psi \circ f \circ \varphi^{-1}(x,y) = \psi(x, S(x)) = (x, S(x) - S(x)) = (x, 0)$$

This completes the proof. ■

3 Group actions on smooth manifolds

We fill the gap in the proof of lens spaces are manifolds.

3.1 Definition

A (left) **group action** G on a set X is a map $\iota : G \to \mathrm{Fun}(X, X)$ such that:

1. $\iota(e) = \mathrm{id}_X$
2. $\iota(gh) = \iota(g) \circ \iota(h), \forall g, h \in G$

If the induced maps on X a manifold are all smooth, we call it a smooth action.

We write gx to mean $\iota(g)(x)$.

The **orbit** of $x \in X$ by G is $G(x) := \{y \in X : \exists g \in G, gx = y\}$.

3.2 Definition

1. We say a group action $\iota : G \to \mathrm{Fun}(X, X)$ is **free**, if $\forall g \neq e, \iota(g) \neq \mathrm{id}_X$.
2. We say a group action $\iota : G \to \mathrm{Fun}(X, X)$ for X a topological space is **properly discontinuous** if for any $y \in X$ not in the orbit $G(x)$, we can find neighbourhoods $x \in U$ and $y \in V$ such that $g(U) \cap V = \emptyset$.

3.3 Theorem

Let X^n be a smooth manifold, let G acts smoothly on X freely and properly discontinuously. Then the orbit space X/G is an n-dimensional smooth manifold.

3.3.1 Proof

We firstly need to show X/G is hausdorff. Let $[x] \neq [x']$ in X/G, hence x' is not in the orbit of x. Now since G acts properly discontinuously, we can find $x \in U$ and $x' \in V$ that separated x and x'. Since π is an open map, $\pi(U)$ and $\pi(V)$ are enough for our purpose(check they are disjoint!).

Second countability is clear.

Pick a point $[x] \in X/G$, then as the action of G is free and properly discontinuous, we can find an open neighbourhood U of x such that $g(U) \cap U = \emptyset, \forall g \neq e$. Keeping shrinking the open to make sure it lies in a chart $\varphi : U \to \mathbb{R}^n$ of X.

Now let $V = \pi(U)$, this is open as π is open and by construction for every point $v \in V$ there's only one unique point $u \in U$ such that $\pi(u) = v$. With some abuses of notation we denote this by π^{-1}(The true preimage has more disjoint copies of U, which is related to the covering space). We set $\psi : V \to \mathbb{R}^n$ by $\psi = \varphi \circ \pi^{-1}$.

The claim is now $\{V, \psi\}_{[x] \in X/G}$ will be a smooth atlas. Clearly it covers X/G. Again since π is a continuous open map and φ is a homeomorphism, we conclude ψ is a homeomorphism. Thus X/G is a topological manifold.

Let $v \in V \cap V'$, take $x \in U$ and $y \in U'$ such that $\pi(x) = \pi(y) = v$, thus there is a $g \in G$ such that $g(y) = x$. We can consider the chart $\left(g(U'), \varphi' \circ g^{-1}\right)$ on X since G is a smooth action. In order to show the transition map $\psi' \circ \psi^{-1}$ is smooth, it is enough to show it is smooth in a neighbourhood of $\psi(v) = \varphi(x)$. On the small open $\varphi(U \cap g(U'))$ we then have

$$\psi' \circ \psi^{-1} = \varphi' \circ g^{-1} \circ \pi^{-1} \circ \left(\varphi \circ \pi^{-1}\right)^{-1} = \left(\varphi' \circ g^{-1}\right) \circ \varphi^{-1}$$

which is smooth. ■

References

[Rei35] K. Reidemeister, Homotopieringe und Linsenräume, **11**, (1935).

[Lee12] J. M. Lee, *Introduction to Smooth Manifolds* (Springer New York, NY, New York, USA, 2012).