Week 0 : Point-Set topology

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We will review some important concepts and theorems in point-set topology.

1. Product topology

In the lecture we have learned:

1.1. Definition

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. The **product topology** on $X \times Y$ is the (unique) topology generated by the basis $\mathcal{B} = \{U \times V : U \in \mathcal{O}_X, V \in \mathcal{O}_Y\}$.

1.2. Remark

Not every open in the product topology looks like $U \times V$ where U, V are opens in X and Y respectively. We can consider the following example:

1.3. Example

It is easy to show that the product topology on $\mathbb{R}^1 \times \mathbb{R}^1$ coincides with the euclidean topology on $\mathbb{R}^2$. Now consider the set $M := \{(x, y) : xy > 1, x > 0\}$. This is open in $\mathbb{R}^2$, but hard to express it as $U \times V$.

If all opens in $X \times Y$ look like $U \times V$, we get the so called **box topology**, which is a little bit coarser than the product topology. But if the product is finite, we can prove that the two topologies are equivalent.

You will know another nice way to characterize the product topology in the exercise sheet:

1.4. Proposition

The product topology on $X \times Y$ is the smallest topology, such that the canonical projections $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are continuous.

1.5. Remark

This construction is common:

1. Let $\sim$ be an equivalent relation on X. The quotient topology is the finest topology on $X/\sim$, such that the canonical surjection $\pi : X \to X/\sim$ is continuous.
2. Let $X \times_Z Y$ denote the set-theorectic pullback of $f : X \to Z$ and $g : Y \to Z$, that is $X \times_Z Y := \{(x, y) \in X \times Y : f(x) = g(y)\}$. Then the pullback topology is the smallest topology on $X \times_Z Y$, such that π_X and π_Y are continuous.
3. Let $f : X \to Y$, Y a topological space. Then the pullback topology associated to f on X is the smallest topology such that f is continuous.

2. Compact spaces

Compactness is a very strong property on spaces. We will see later that compactness implies many good properties: solvability of flows etc.

A compact manifold without boundary is called closed.

2.1. Proposition

Let X be a metric space, then the followings are equivalent:

1. X is cover-compact.
2. X is sequence-compact.
3. X is complete and for every $\varepsilon > 0$ we have finitely many $x_1, ..., x_n$ such that $X = \bigcup_{i=1}^{n} B_\varepsilon(x_i)$.

If one of the condition is met, we call the space X quasi-compact.

2.1.1. Proof

TODO

2.2. Remark

Why quasi compact? Traditionally when people talked about compact spaces, they often meant quasi-compact and hausdorff spaces, i.e. T_3-separable. Even in modern algebraic geometry people reserve the name “compact” for other meanings.

2.3. Proposition

If X and Y are quasi-compact, then the space $X \times Y$ with product topology is also quasi-compact.

2.3.1. Proof

Let $\{U\}_{\lambda \in \Lambda}$ be an open cover of $X \times Y$. For each $(x, y) \in X \times Y$, we can choose some $\lambda = \lambda(x, y)$ such that $(x, y) \in U_{\lambda(x,y)}$. By the definition of product topology the point (x, y) is contained in some open box $M_{x,y} \times N_{x,y} \in U_{\lambda(x,y)}$

Now fix x and vary y. We observe that the collection of sets $\left\{N_{x,y}\right\}_{y \in Y}$ is an open cover of $\{x\} \times Y \cong Y$. Since by assumption Y is quasi-compact, we can find a finite cover $\left\{N_{x,y_i}\right\}_{j=1}^{n_x}$ of $\{x\} \times Y$.

Let $M_x = \bigcap_{j=1}^{n_x} M_{x,y_j}$. Since M_x is the intersection of finitely many open sets, it is itself open. Since X is quasi-compact, there are finitely many x_is such that $\left\{M_{x_i}\right\}_{i=1}^{m}$ forms an open cover of X. The collection of sets $\left\{U_{\lambda(x_i,y_j)}\right\}_{i=1,j=1}^{m,n_{x_i}}$ is a finite cover of $X \times Y$, hence quasi-compact. ■

3. Lusternik–Schnirelmann category

How many charts in general do we need in order to cover a topological manifold? It happens to be $n + 1$ where n is the dimension of the manifold. This invariant is so called Lusternik–Schnirelmann category, it has a deep connection with Morse theory.

4. Solutions

4.1. Exercise 1

4.1.1. Proof of a)

1. The empty set is open in M open and hence in X too as the intersection with X. X is open as $X = M \cap X$.
2. Let U_1, U_2 be opens in X, then we have $U_1 = V_1 \cap X$ and $U_2 = V_2 \cap X$ where V_1, V_2 are opens in M by definition. And
$$U_1 \cap U_2 = (V_1 \cap V_2) \cap X$$
Since $V_1 \cap V_2$ is open in M, we know that $U_1 \cap U_2$ is open in X.
3. Let $\{U_i = V_i \cap X\}_{i \in I}$ be a family of opens in X where $\{V_i\}_{i \in I}$ are open in M, then
$$\bigcup_{i \in I} U_i = \bigcup_{i \in I} (V_i \cap X) = \left(\bigcup_{i \in I} V_i \right) \cap X$$
Since $\bigcup_{i \in I} V_i$ is open in M, the union of U_is are open in X. ∎

4.1.2. Proof of b)

Suffices to show $\mathcal{O}_{\mathbb{R}^1 \times \{0\}}$ is generated by open balls in the standard topology $\mathcal{O}_{\text{eucl}}$. In 1-dimensional case they are just open intervals. Let $M = U \cap (\mathbb{R}^1 \times \{0\})$ be a open set in subspace topology. Then since U is open in 2-dimensional standard topology, for every $x \in M \subset U$, we have $B_{\varepsilon(x)}(x) \subset U$. But
$$B_{\varepsilon(x)} \cap (\mathbb{R}^1 \times \{0\}) = (x - \varepsilon(x), x + \varepsilon(x)) \subset M$$
we thus have $M = \bigcup_{x \in M} (x - \varepsilon(x), x + \varepsilon(x))$ generated by open intervals. ∎

4.2. Exercise 2

4.2.1. Proof of a)

Let $A \subset M$ be a closed subset, let $\{U_i\}_{i \in I}$ be an open cover of A, where U_is are in M open. Then as A is closed, $M - A$ is open in M. We construct a open cover of M by
$$M = (M - A) \cup \bigcup_{i \in I} U_i$$
By the compactness of M we can find a finite cover of M:
$$M = (M - A) \cup \bigcup_{j=1}^{n} U_j$$
Thus $\{U_j\}_{j=1}^{n}$ is a finite subcover of $\{U_i\}_{i \in I}$ and A is compact.

4.2.2. Proof of b)

Let $K \subset M$ be a compact subset and M is Hausdorff. In order to show K is closed in M, it suffices to show $U := M - K$ is open. Take $x \in U$, for every $y \in K$ we have $x \neq y$. As M is Hausdorff, we can find opens $x \in U_y$ and $y \in V_y$ that separates two points. Consider $K = \bigcup_{y \in K} V_y$ an open cover, since K is compact, we know there is a finite cover $K = \bigcup_{i=1}^n V_{y_i}$. Set $\tilde{U} := \bigcap_{i=1}^n U_{y_i}$, by definition, this is an open neighbourhood of x contained in U and since x is arbitrary, U is open. ■

4.2.3. Examples of c)

1. Take $\{x, y\}$ two points with indiscrete topology, we have $\{x\}$ finite and compact, but $\{x\}$ is not closed in $\{x, y\}$ as the whole space not Hausdorff.

2. Take the line with two zeroes. Every closed interval containing just one zero is compact but not closed.

4.3. Exercise 3

4.3.1. Proof of a)

If not, then we may find two non-empty disjoint clopen subsets U and V such that $X = U \sqcup V$. Let $x \in U$ and $y \in V$. Since X is Z-connected, we can find a continuous function $f : Z \to X$ such that $x, y \in f(Z)$. By assumption Z is connected, thus $f(Z)$ is also connected. Now we construct two non-empty disjoint clopen subsets in $f(Z)$ that contradicts this fact.

Let $U' := f(Z) \cap U$ and $V' := f(Z) \cap V$. Clearly $f(Z) = U' \sqcup V'$ and they are non-empty because x and y live in there respectively. U' and V' are also clopen by the definition of subspace topology and the fact that they form a cover of $f(Z)$, this shows $f(Z)$ is not connected, contradiction! ■

4.3.2. Example of b)

An equivalent example: $Z = \mathbb{S}^1$.

Somewhat weaker: take Z to be the topological sinus curve, this is a connected but not path-connected space.

Somewhat stronger: set Z as $\mathbb{D}^2 := \overline{B_1(0)} \subset \mathbb{R}^2$, this has some problems with the fundamental group of the space X.