Week 3: Tangent spaces and vector bundles

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1. Construct vector bundles

1.1. Pullback bundles

Pullback is such an important construction in general mathematics, that if people have constructed a new mathematical object, one of the first natural question would be: "How does it behave under pulling back?"

Luckily, the pullback of vector bundle is still a vector bundle.

Let $f : X \to Y$ smooth, $\pi : E \to Y$ a vector bundle on Y, then we can define the pullback bundle f^*E as

$$\begin{tikzcd}
f^*E \arrow[r] \arrow[d] & E \arrow[d, "\pi"'] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

By exercise 1 of sheet 2, this is indeed a vector bundle on X.

A nice property of pullback is that it preserves the direct sum, i.e.

$$f^*(E_1 \oplus E_2) = f^*(E_1) \oplus f^*(E_2)$$

1.2. Tensor product

In the categorical language, pullback is a limit and tensor product is a colimit. The category of vector spaces on a field k admits finite limits and colimits, this leads us to the following construction.

Consider E_1 and E_2 two vector bundles on the same base space B. Like the Whitney sum, under finer trivialization covers the fiberwise tensor product $E_{1,p} \otimes E_{2,p}$ induces a global vector bundle $E_1 \otimes E_2$, the tensor product of vector bundles.

For line bundles, there is even a simpler discription: Consider the transition maps g_{ij} and g'_{ij} of E_1 and E_2 respectively under suitable restriction, the unique vector bundle $E_1 \times E_2$ is given by transition functions $g_{ij} \cdot g'_{ij}$.

With this construction we can easily prove the following cool fact:

1.2.1. Proposition

Let $M \to \mathbb{S}^1$ be the Möbius bundle, then we have $M \otimes M \cong \mathbb{S}^1 \times \mathbb{R}$.

Proof: As Möbius bundle has transition maps $g_{ij} = \pm 1$, the tensor product has transition maps $(\pm 1)^2 = 1$ everywhere, hence trivial. ∎

1.3. Classification of vector bundles

The above proposition motivates us to classify line bundles on a manifold. As by construction, the isomorphic classes of line bundles form an abelian group with $\otimes$ the multiplication and $M \times \mathbb{R}$ as unit. This group is called **Picard group** of a manifold M and denoted by $\mathrm{Pic}(M)$.

1.3.1. Example

1. The Picard group $\mathrm{Pic}(\mathbb{S}^1)$ is isomorphic to $\mathbb{Z}/2$, i.e. the Möbius bundle is the only non-trivial generator.
2. The Picard group of $\mathbb{S}^2 \cong \mathbb{CP}^1$ is, however, much more interesting. One can show that $\mathrm{Pic}(\mathbb{CP}^1) \cong \mathbb{Z}$ with the so called **Serre twisting** as the generator.

A more ambitious goal is to characterize all vector bundles at the same time. Frank Adams did this via K-theory as a cohomology theory.

The isomorphic classes of vector bundles form a commutative monoid under the Whitney sum and $M \times *$ as a unit. Using the Grothendieck group completion, it turns to a group, called the 0-th K-group $K^0(X)$ of the space X.

A similar spirit exists for $K^1(X)$ the first K-group, but the story becomes a little bit trickier.

A milestone for 20th century topology is the following theorem:

1.3.2. Theorem(Bott periodicity)

Let X be a locally compact Hausdorff space(i.e. a manifold), then the complex K-theory($\mathbb{C}$-vector bundles) has period 2 and the real K-theory has period 8, precisely

$$K^{n+2}(X) \cong K^n(X)$$
$$K^{n+8}_{\mathbb{R}}(X) \cong K^n_{\mathbb{R}}(X)$$

Though the K-theory can be formulated in a pure algebraic setting, this periodicity is unique for topological K-theory.

2. Parallelizable tangent bundles

2.1. Product and parallelizablity

The product manifold is compatible with parallelizablity in the sense of following theorem.

2.1.1. Theorem

Let M_1 and M_2 be two parallelizable manifold, then the product $M_1 \times M_2$ is also parallelizable.

Proof: follows directly from the fact $T_{(p,q)}(M_1 \times M_2) \cong T_p M_1 \oplus T_q M_2$. ■

2.1.2. Corollary

The n-torus $\mathbb{T}^n = \mathbb{S}^1 \times ... \times \mathbb{S}^n$ is parallelizable

2.2. Parallelizable spheres

We have seen in the lecture that $\mathbb{S}^1$ is parallelizable and the linearly independent section is given by $J \cdot (x, y)$ where $(x, y) \in \mathbb{S}^1 \subset \mathbb{R}^2$ and $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. This matrix can be identified in the sense of complex numbers to be i(both are rotation of 90 degrees).

The astonishing fact is that $\mathbb{S}^3 \subset \mathbb{R}^4$ is also parallelizable with the linearly independent sections given by the structure of quaternion $\mathbb{H}$.

$$s_1(x) := x \cdot i = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} x$$

$$s_2(x) = x \cdot j = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} x$$

$$s_3(x) = x \cdot k = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} x$$

The same works fine for $\mathbb{S}^7 \subset \mathbb{R}^8$ with octonion $\mathbb{O}$. Also $\mathbb{S}^0 \subset \mathbb{R}$ is parallelizable by $1 \in \mathbb{R}$.

The $\mathbb{R}$-algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ are all division algebras, which means fields without commutativity on multiplication assumed. This one-to-one correspondence of $\mathbb{R}$-division algebras with parallelizable spheres has an end, formulated in following two theorems.

2.2.1. Corollary

The only parallelizable spheres are $\mathbb{S}^0, \mathbb{S}^1, \mathbb{S}^3$ and $\mathbb{S}^7$.

2.2.2. Corollary

The only $\mathbb{R}$-division algebras are $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$.

The bridge to relate these two deep results from topological world and algebraic world is the following. Remember we have known in the lecture the Hopf fiber sequence:

$$\mathbb{S}^{2n-1} \to \mathbb{S}^{4n-1} \to \mathbb{S}^{2n}$$

2.2.3. Theorem(Hopf invariant one)

Suppose there is a continuous map $f : \mathbb{S}^{4n-1} \to \mathbb{S}^{2n}$ with Hopf invariant $h(f) = 1$, then $n = 1, 2$ or 4.

The proof uses K-theory and the famous Adams operations on spheres.

3. The meaning of tangent spaces

3.1. Differentiating functions

One of the most important function of tangent vectors is giving a canonical way to differentiate functions on manifolds. Consider $f : M \to \mathbb{R}$ smooth, we can view a tangent vector as a derivation: $v(f) := D_p f(v)$ in a local coordinate. If we have a vector field X that varies smoothly, then again we get a smooth derivative of f as $X(f) : M \to \mathbb{R}, p \mapsto X_p(f) := D_p f\left(X_p\right)$.

3.2. Measuring geometry

Moreover if we equip a inner product on the tangent space and let it vary smoothly, we can also measure many geometric properties. This inner product is so called Riemann metric on manifolds and it truns out every smooth manifold can be equipped with a Riemann geometry.

Different metrics induce different geometries. For example, the metric of $\mathbb{T}^2 \subset \mathbb{R}^3$ induced from standard euclidean metric is different from the product metric from $\mathbb{S}^1$ to $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$! The later one is called **flat torus**(or **Clifford torus**) and cannot be embedded into $\mathbb{R}^3$.

3.2.1. Definition

The **length** of a smooth curve $\gamma : (a, b) \to M$ is defined to be

$$l(\gamma) := \int_a^b \|\gamma'(t)\| dt = \int_a^b \sqrt{\langle \gamma'(t), \gamma'(t) \rangle} dt$$

it's invariant under reparametrization.

To measure the volume we need to know differential forms first.