Tame homotopy type, stratified

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1. Étale and tame homotopy type

1.1 Problem in char p

For X/k a coherent scheme such that $X_{\overline{k}}$ is connected, there is an exact sequence

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1-truncation of $\Pi_\infty^{\text{ét}}(X)$ recovers the étale fundamental group of X.

We have a refinement of the fundamental sequence:

$$\Pi_\infty^{\text{ét}}(X_{\overline{k}}) \to \Pi_\infty^{\text{ét}}(X) \to \mathrm{B}G_{\overline{k}/k}.$$

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$$\pi_1^t(X) := \lim_{\overrightarrow{Y}} \mathrm{Aut}(Y/X).$$

We can also define the tame site X_t of a scheme X and the tame homotopy type $\Pi_\infty^t(X)$ is the associated profinite shape of the topos $\mathrm{Shv}_t(X)$.

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We focus on the rigidity.

2. Stratification

2.1 Log geometry

Suppose $j : X \hookrightarrow \overline{X}$ is a good compactification. The boundary defines a logarithmic structure on $\overline{X}$.

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A purity result allows us to identify finite Kummer étale covers of $\left(\overline{X}, D\right)$ with finite étale covers of X “tame” along D.

2.2 Stratified tame topos

Let X/S be a fs log scheme (or of type V_{div}) such that the underline scheme $\underline{X}$ is coherent.

The topos $\mathcal{X}_{\text{két}} := \mathrm{Shv}_{\text{két}}(X)$ is stratified under the spectral space $\widetilde{X^{\mathrm{Zar}}} := \mathrm{Shv}(\mathrm{Open}(\underline{X}^{\mathrm{Zar}}))$.

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By taking Kummer étale cover $Y \to X$ that is tame relative to S, we get a topos $\mathcal{X}_t := \mathrm{Shv}_{\mathrm{t}}(X) \subset \mathrm{Shv}_{\text{két}}(X)$, again stratified.

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Suppose there is a global chart $\alpha : P \to \mathcal{M}(X)$ for P a fs integral monoid, then we have a continuous map

$$\overline{\alpha} : X^{\mathrm{Zar}} \to \mathrm{Spec}(P), x \mapsto P - \alpha_x^{-1}(\mathcal{O}_{X,x}^{\times}).$$

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Under some finiteness condition, the induced morphism stratifies the associated topos:

$$\mathcal{X}_t \to \widetilde{X^{\text{Zar}}} \overset{\overline{\alpha}_*}{\to} \widetilde{\text{Spec}(P)}.$$

3. Structure of $\Pi^t_\infty(X)$

3.1 Rigidity

With the above formulation we are able to prove:

Theorem 3.1.1 Let k be a separably closed field and $k \subset K$. Let X/k be a coherent scheme admits a good compatification, then we have an equivalence of pro-animae

$$\Pi^t_\infty(X) \cong \Pi^t_\infty(X_K).$$

3.1 Rigidity

The proof uses a rigidity of tame cohomology together with the following relation:

$$H^1_t(X, \mathbb{Z}/p) \cong \mathrm{Hom}_{\mathrm{cont}}(\pi_1^t(X), \mathbb{Z}/p).$$

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$$H^1_t(X, \mathbb{Z}/p) \cong \mathrm{Hom}_{\mathrm{cont}}(\pi^t_1(X), \mathbb{Z}/p).$$

Theorem 3.1.2 Let k be a separably closed field of characteristic p. Let $k \subset K$, then for any coherent scheme X/k that admits a good compatification we have

$$R\Gamma_t(X, \mathbb{Z}/p(i)) \simeq R\Gamma_t(X_K, \mathbb{Z}/p(i)).$$

where the theorem relies on the rigidity of syntomic cohomology.

3.2 Speculations

Conjecture 1 (Exodromy for p-adic sheaves): Let X/S be a coherent fs log scheme with some finiteness condition, $\mathrm{char}(S) = p$, then there is a category $\mathrm{Gal}^t(X)$ such that

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Conjecture 2 (Affine(?) $K(\pi, 1)$): Under which condition on the connected scheme X/k over an algebraically closed field is $\Pi^t_\infty(X)$ 1-truncated?

Thank you for listening!