Talk 6: Viazovska's Ansatz

In this talk we will explain the motivation and methods of Viazovska's construction for the magic function of E_8 sphere packing as in [7].

Recall conceptually we want radial Fourier eigenfunctions a and b of eigenvalues $+1$ and -1 that satisfy following conditions:

$$\begin{aligned} a &: \mathbb{R}^8 \to \mathbb{C}, a \in \mathcal{S}(\mathbb{R}^8) \\ a(|x|) &= 0, \forall |x| = \sqrt{2n}, n \geq 1 \\ a'(|x|) &= 0, \forall |x| = \sqrt{2n}, n \geq 2 \\ \hat{a}(x) &= a(x) \end{aligned} \tag{0.1}$$

and

$$\begin{aligned} b &: \mathbb{R}^8 \to \mathbb{C}, b \in \mathcal{S}(\mathbb{R}^8) \\ b(|x|) &= 0, \forall |x| = \sqrt{2n}, n \geq 1 \\ b'(|x|) &= 0, \forall |x| = \sqrt{2n}, n \geq 2 \\ \hat{b}(x) &= -b(x) \end{aligned} \tag{0.2}$$

This is due to the theorem of linear programming bound of Cohn and Elkies in [3].

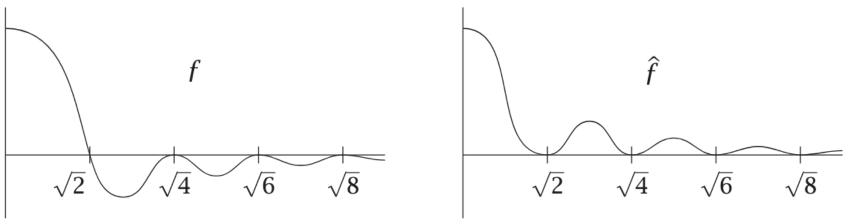

Figure 1: Schematic plot from [1] showing roots behavior of magic functions

1 Gaussian and its transformation

Definition 1.1 The complex Gaussian is a radial Schwartz function

$$\mathcal{G} : \mathbb{R}^d \to \mathbb{C}, x \mapsto e^{\pi i z |x|^2}. \tag{1.1}$$

Lemma 1.2 The Fourier transformation of complex Gaussian of dimension d is given by:

$$\hat{\mathcal{G}}(y) = \left(\frac{i}{z}\right)^{\frac{d}{2}} e^{\pi i(-1/z)|y|^2} \tag{1.2}$$

.

Proof. Notice we have $\mathcal{F}\left(e^{-\pi x^2}\right)(y) = e^{-\pi y^2}$ for $x \in \mathbb{R}$. We calculate

$$
\begin{aligned}
\hat{\mathcal{G}}(y) &= \int_{\mathbb{R}^d} e^{\pi i z|x|^2} e^{-2\pi i x\cdot y} dx \\
&= \prod_{i=1}^{d} \int_{-\infty}^{\infty} e^{\pi i z x_i^2} e^{-2\pi i x_i y_i} dx_i \\
&= \left(-\frac{1}{zi}\right)^{\frac{d}{2}} \prod_{i=1}^{d} \int_{-\infty}^{\infty} e^{-\pi x_i^2/(-z)} e^{-2\pi i x_i y_i} dx_i \\
&= \left(\frac{i}{z}\right)^{\frac{d}{2}} e^{\pi i(-1/z)|y|^2}.
\end{aligned}
\tag{1.3}
$$

■

2 Zeroes at lattice points

2.1 Fourier coefficients

For a 2-periodic holomorphic function $\psi : \mathbb{H} \to \mathbb{C}$ with Fourier series

$$\psi(z) = \sum_{n\in\mathbb{Z}} a_n e^{\pi i n z} \tag{2.1.1}$$

we have

$$a_n = \int_0^1 \psi(t) e^{-\pi i n t} dt. \tag{2.1.2}$$

where $\{e^{\pi i n z}\}_{n\in\mathbb{Z}}$ is an L^2-orthogonal basis of Fourier transformations.

This leads us to define

$$g(x) := \frac{1}{2}\int_{-1}^{1} \psi(z)\mathcal{G}(x) dz \tag{2.1.3}$$

as the contour integral along upper circle in $\mathbb{H}$. And we should have $g(\sqrt{n}) = a_{-n}(\psi)$. Thus, in order to control the behavior of zeroes at lattice points, it suffices to control the Fourier coefficients of a periodic function on $\mathbb{H}$.

Using Lemma 1.2 we can calculate the Fourier transformation of $g(x)$ for $d = 8$ as follows:

$$
\begin{aligned}
\hat{g}(y) &= \frac{1}{2}\int_{-1}^{1} \psi(t)\left(\frac{i}{z}\right)^{\frac{d}{2}} e^{\pi i(-1/z)|y|^2} dt \\
&= \frac{1}{2}\int_{-1}^{1} \psi(t) z^{-4} e^{\pi i(-1/z)|y|^2} dt \\
&= -\frac{1}{2}\int_{-1}^{1} \psi(-1/u) u^2 e^{\pi i u|y|^2} du.
\end{aligned}
\tag{2.1.4}
$$

If we want g to be a Fourier eigenfunction, say of eigenvalue -1, then this is amount of saying $\psi(-1/z) = z^{-2}\psi(z)$, using slash operator this is saying

$$\psi|_{-2}S = \psi. \tag{2.1.5}$$

Thus, it is natural to consider some modular forms in order to construct these magic functions.

The zeroes condition of $g(x)$ implies, that ψ has no Fourier coefficients less that -1, and since $g(x)$ must be a Schwartz function, for $|x|$ sufficiently large, we have $g(x) \to 0$, this is amount of saying ψ vanishes at ± 1, together we have

$$\begin{aligned} \psi(it) &= a_{-1}e^{\pi t} + a_0 + O(e^{-\pi t}) \\ \psi(-1/t+1)t^2 &= \sum_{n=1}^{\infty} a_n e^{\pi i n t}. \end{aligned} \tag{2.1.6}$$

2.2 Laplace transformation

We write our candidate $g(x)$ in a more convenient form, so that the integral only depends on real parameters. We do the calculation:

$$\begin{aligned} g(x) &= \frac{1}{2}\int_{-1}^{1} \psi(z)e^{\pi i z|x|^2}dz \\ &= \frac{1}{2}\int_{-1}^{i} \psi(z)e^{\pi i z|x|^2}dz - \frac{1}{2}\int_{1}^{i} \psi(z)e^{\pi i z|x|^2}dz \\ &= \frac{1}{2}\int_{-1}^{-1+iR} \psi(z)e^{\pi i z|x|^2}dz + \frac{1}{2}\int_{-1+iR}^{1+iR} \psi(z)e^{\pi i z|x|^2}dz - \frac{1}{2}\int_{1}^{1+iR} \psi(z)e^{\pi i z|x|^2}dz \\ &= \frac{1}{2}\int_{-1}^{-1+i\infty} \psi(z)e^{\pi i z|x|^2}dz - \frac{1}{2}\int_{1}^{1+i\infty} \psi(z)e^{\pi i z|x|^2}dz. \end{aligned} \tag{2.2.1}$$

where we use the vanishing condition (2.1.6) to see that

$$\int_{-1+iR}^{1+iR} \psi(z)e^{\pi i z|x|^2}dz \to 0 \text{ for } R \to \infty \tag{2.2.2}$$

for all $|x| \geq \sqrt{2}$.

As $\psi(z)$ is 2-periodic, then in particular, it is symmetric at the two sides of imaginary line and $\psi(it-1) = \psi(it+1)$, hence we rewrite (2.2.1) as

$$g(x) = \frac{e^{-\pi i|x|^2} - e^{\pi i|x|^2}}{2}\int_{0}^{i\infty} \psi(u+1)e^{\pi i u|x|^2}du \tag{2.2.3}$$

by change of variables $u = z+1$.

Definition 2.2.1 Given an L^1-function ψ on $\mathbb{H}$, its Laplace transformation is defined by

$$\mathcal{L}(\psi)(s) := \int_0^\infty \psi(t) e^{-st} dt. \tag{2.2.4}$$

Thus our candidate for eigenfunction is given by the Laplace transformation of a modular forms like functions times $\sin(\pi|x|^2)$:

$$g(x) = \sin(\pi|x|^2) \int_0^\infty \psi(it+1) e^{-\pi t|x|^2} dt. \tag{2.2.5}$$

3 Contours and functional equations

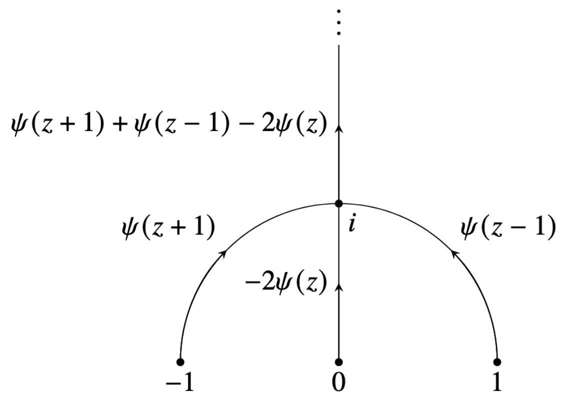

Figure 2: The pitch fork like contour integral from [2].

In this section we deduce the real functional equations satisfied by modular forms depending on (0.1) and (0.2). We assume $|x|$ is sufficiently large to bound the integral.

To match our period we consider the function

$$g(x) = \sin(\pi|x|^2/2)^2 \int_0^\infty \psi(it) e^{-\pi t|x|^2} dt \tag{3.1}$$

we use the squared sine function to give double zeroes at lattice points, instead of single zero. To cancel the double zero at $\sqrt{2}$ and triple zeroes at 0, we need to manipulate ψ so that it has a pole of order 4 at 0 and a simple pole at $\sqrt{2}$. This function can be split into the integrals with respect to the above figure:

$$\begin{aligned} &2\int_0^{i\infty} \psi(it)e^{-\pi t|x|^2}dt - \int_{-1}^{-1+i\infty} \psi(it+1)e^{-\pi t|x|^2}dt - \int_1^{1+i\infty} \psi(it-1)e^{-\pi t|x|^2}dt \\ &= 2\int_0^i \psi(it)e^{-\pi t|x|^2}dt - \int_1^i \psi(it-1)e^{-\pi t|x|^2}dt - \int_{-1}^i \psi(it+1)e^{-\pi t|x|^2}dt \\ &\quad + \int_i^\infty (2\psi(it) - \psi(it-1) - \psi(it+1))e^{-\pi t|x|^2}dt. \end{aligned} \tag{3.2}$$

The Fourier transformation of $g(x)$ is given by

$$
\begin{aligned}
&2\int_0^i \psi(it)(it)^{-4}e^{-\pi|x|^2/t}dt - \int_1^i \psi(it-1)(it)^{-4}e^{-\pi|x|^2/t}dt - \int_{-1}^i \psi(it+1)(it)^{-4}e^{-\pi|x|^2/2}dt \\
&\qquad + \int_i^\infty (2\psi(it) - \psi(it-1) - \psi(it+1))(it)^{-4}e^{-\pi|x|^2/t}dt \\
&= -2\int_i^{i\infty} \psi(-1/u)u^2 e^{\pi iu|x|^2}du - \int_{-1}^i \psi(-1/u-1)u^2e^{\pi iu|x|^2}du - \int_1^i \psi(-1/u+1)u^2e^{\pi iu|x|^2}du \\
&\qquad -2\int_0^i (\psi(-1/u) - \psi(-1/u+1) - \psi(-1/u-1))u^2e^{\pi iu|x|^2}du.
\end{aligned}
\tag{3.3}
$$

where the equation is a consequence of change of variables $it \mapsto -1/it$.

We want $\hat{g}(x) = -g(x)$, hence by comparing integrals we must have

$$
\begin{cases}
\psi\ |_{-2}\ TS = -\psi|_{-2}T^{-1} \\
\psi\ |_{-2}\ T^{-1}S = -\psi|_{-2}T \\
2\psi\ |_{-2}\ S = 2\psi - \psi\ |_{-1}T - \psi|_{-1}T^{-1}
\end{cases}
\tag{3.4}
$$

This is equivalent to

$$
\begin{cases}
\psi\ |_{-2}TS = -\psi\ |_{-2}T \\
\psi\ |_{-2}T^2 = \psi \\
\psi|_{-2}\ T + \psi|_{-2}\ S = \psi
\end{cases}
\tag{3.5}
$$

Note the third applies the second by applying $|_{-2}S$ on both sides.

Upshot is, ψ should be a meromorphic modular form of weight -2 of $\Gamma(2)$ that satisfying

$$
\psi|_{-2}\ T + \psi|_{-2}\ S = \psi. \tag{3.6}
$$

We can multiply ψ by Δ to get a holomorphic modular form of weight 10 of $\Gamma(2)$. Note that the space $\mathcal{M}_{10}(\Gamma(2))$ is generated by

$$
\left\{\theta_{01}^{20-4j}\theta_{10}^{4j}\right\}_{j=0}^{5} \tag{3.7}
$$

a basis of 6 elements. where $\theta_{01}(z) = \sum_{n\in\mathbb{Z}}(-1)^n e^{\pi in^2z}$ and $\theta_{10}(z) = \sum_{n\in\mathbb{Z}} e^{\pi i(n+\frac{1}{2})^2z}$.

In order to solve the given functional equation, one can use a computer algebra system, see e.g. [[5], Appendix B].

Similarly if we want a Fourier eigenfunction of eigenvalue 1. We should have

$$
\begin{cases}
\psi\ |_{-2}\ TS = \psi|_{-2}T^{-1} \\
\psi\ |_{-2}\ T^{-1}S = \psi|_{-2}T \\
2\psi\ |_{-2}\ S = -2\psi + \psi\ |_{-1}T + \psi|_{-1}T^{-1}
\end{cases}
\tag{3.8}
$$

As $(ST)^3 = I$ we see this is equivalent to

$$\begin{cases} \psi \mid_{-2} ST = \psi|_{-2}S \\ 2\psi \mid_{-2} S = -2\psi + \psi \mid_{-1} T + \psi|_{-1}T^{-1} \end{cases} \tag{3.9}$$

Suppose we set $\chi := \psi|_{-2}S$, we see the second equation tells us $\chi|_0 S = \chi$ and the first says $\chi|_0 T = \chi$, this leads us to search for a meromorphic modular form of weight 0 of $\Gamma(1)$. However, this linear equation is not solvable in our given basis of modular forms. We need instead quasimodular forms.

Definition 3.1 An almost modular form of weight k of Γ is a polynomial $f(z) = \sum_{j=1}^{r} f_j(z)y^{-j}$ where $y = \mathrm{im}(z)$ such that:
1. $f|_k\gamma = f, \forall \gamma \in \Gamma$.
2. Each f_j is a holomorphic function on $\mathbb{H} \cup \mathbb{QP}^1$.

The constant r is called the depth of an almost modular form. The holomorphic term of an almost modular form is called a quasimodular form.

Proposition 3.2 [[5], Corollary 3.10] The Eisenstein series E_2, E_4, E_6 form a basis of quasimodular forms.

4 Analytic continuation

Our integral is only well defined as $|x| > \sqrt{2}$, to make it a holomorphic function on $\mathbb{R}^8$, we need to do the analytic continuation as following:

Suppose after searching our ψ has Fourier series

$$\psi(it) = a_{-2}e^{2\pi t} + a_{-1}e^{\pi t} + a_0 + O(e^{-\pi t}) \tag{4.1}$$

then we can remove the divergent term by hand and define $a(x)$ globally as

$$\begin{aligned} a(x) &= \sin(\pi|x|^2/2)^2 \int_0^\infty (a_{-2}e^{2\pi t} + a_{-1}e^{\pi t} + a_0)e^{-\pi t|x|^2}dt \\ &+\sin(\pi|x|^2/2)^2 \int_0^\infty (\psi(it) - a_{-2}e^{2\pi t} - a_{-1}e^{\pi t} - a_0)e^{-\pi t|x|^2}dt \\ &= \sin(\pi|x|^2/2)^2 \left(\frac{a_{-2}}{\pi(|x|^2-2)} + \frac{a_{-1}}{\pi(|x|^2-1)} + \frac{a_0}{\pi|x|^2} \right) \\ &+\sin(\pi|x|^2/2)^2 \int_0^\infty (\psi(it) - a_{-2}e^{2\pi t} - a_{-1}e^{\pi t} - a_0)e^{-\pi t|x|^2}dt \end{aligned} \tag{4.2}$$

and $a(x)$ has removable singularities as $|x| = 0, 1$ and $\sqrt{2}$.

Remark 4.1 Note that in [7] Viazovska defined the eigenfunction firstly by splitting apart integrals and showed this can be represented as a Laplace transformation for $|x| > \sqrt{2}$.

5 Putting pieces together

There are still two things rest for compiling a magic function of our purpose.

Firstly, we need to normalize the linear combination of the eigenfunctions a and b, so that $\hat{f}(0) = f(0) = 1$. Since our functions a and b take value in $i\mathbb{R}$, we should make

$$f(x) = Aia(x) + Bib(x). \tag{5.1}$$

It turns out $b(0) = 0$ thus A is solely determined to normalize f.

We need to check now whether

$$\begin{aligned} f(x) &\geq 0, \forall |x| \geq \sqrt{2} \\ \hat{f}(x) &\leq 0, \forall x \in \mathbb{R}^8. \end{aligned} \tag{5.2}$$

This should be enough to determine B as a factor for eliminating our $e^{2\pi t}$ term in $a(x)$. The estimation is tricky and was given in the original proof [7] of Viazovska using interval arithmetic. However, we now have a algebraic proof [6] due to Seewoo Lee.

6 Dimension 24 case

The dimension 24 case is similarly to our Ansatz. In fact, only 2 months after the initial proof of dimension 8 case, a proof of dimension 24 case was constructed in [4].

There are two differences in dimension 24:

1. The Leech lattice has no vector length of $\sqrt{2}$, hence, we want a zero at all $|x| \geq 2$ and double zeroes at $|x| \geq \sqrt{6}$.
2. The $z^{\frac{d}{2}-2}$ factor after change of variable will be 10, making it a meromorphic modular form of weight -10. Hence, we need Δ^2 as a divisor.

The estimation part of dimension 24 case is much more tricky as in dimension 8. The reason is that, before analytic continuation, the function

$$A\varphi\left(\frac{i}{t}\right)t^{10} + B\psi(it) \tag{6.1}$$

has a non-zero term at $e^{\pi t}$, making the integral only converge for $|x| > \sqrt{2}$. This can be fixed again by interval arithmetic and Sturm's theorem. In [4], they used a computer algebra system to calculate q^{50} and approximated π with its 10th digits!

Bibliography

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