Cyclotomic structure on THH

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Roadmap:

First we introduce some preliminaries on spectra with G-actions. We may assume G be a finite group or a compact topological group (Farrell-Tate). Then we specialize this to our primary example $\mathbb{T}$, the circle group and its finite cyclic subgroup C_p. This allows us to define the cyclotomic spectra and make a few first examples, including the import exact sequence.

Finally we turn our attention to THH, explaining the so called Tate construction and equip THH with a cyclotomic structure. At this point, we are ready to define TC. We give an introduction of Tate-valued Frobenius and its relation to our THH Frobenii. This is the first step of calculating THH of KU.

1 The Tate construction on equivariant spectra

Definition 1.1 Let $\mathcal{C}$ be an ∞-category and G a finite group. Suppose $\mathcal{C}$ has limits and colimits indexed by $\mathrm{B}G$, then

1. The homotopy orbit functor is given by

$$(\cdot)_{hG} : \mathcal{C}^{\mathrm{B}G} \to \mathcal{C}, F \mapsto \mathrm{colim} F. \tag{1.1}$$

2. The homotopy fixed point functor is given by

$$(\cdot)^{hG} : \mathcal{C}^{\mathrm{B}G} \to \mathcal{C}, F \mapsto \mathrm{lim} F. \tag{1.2}$$

Our goal is to construct a natural transformation $Nm_G : (\cdot)_{hG} \to (\cdot)^{hG}$, mimicking the norm map in Tate cohomology of abelian groups. This needs us to apply some restrictions on the ∞-category $\mathcal{C}$. Generally we are in the following situation.

Construction 1.2 Let $\mathcal{C}$ be a preadditive category. Let $f : X \to Y$ be a map of Kan complexes. We write $f_! : \mathcal{C}^X \to \mathcal{C}^Y$ resp. $f_* : \mathcal{C}^X \to \mathcal{C}^Y$ as the left resp. right Kan extension along f. Let $\delta : X \to X \times_Y X$ be the diagonal. Suppose there is a natural transformation $Nm_\delta : \delta_! \to \delta_*$ and is an equivalence. Consider the pullback diagram

$$\begin{tikzcd}
X \times_Y X \arrow[r, "p_1"] \arrow[d, "p_0"'] & X \arrow[d, "f"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

The composition

$$p_0^* \to \delta_* \delta^* (p_0)^* = \delta_* \overset{Nm_\delta^{-1}}{\to} \delta_! = \delta_! \delta^* (p_1)^* \to p_1^* \tag{1.3}$$

is adjoint to a map $\mathrm{id}_{\mathcal{C}^X} \to p_{0,*}p_1^*$. By [[Lur09], Lemma 6.1.6.3] the right side is equivalent to f^*f_*, this is again adjoint to $f_! \to f_*$, our desired natural transformation.

Thus the core assumption would be the existence of the equivalence Nm_δ. We do induction on the connectedness of f. Suppose f is (-1)-truncated, then $\delta : X \to X \times_Y X$ is an equivalence and so is Nm_δ. Suppose f is 0-truncated and all fibers of f are finite, then δ is (-1)-truncated, by [[Lur09], Proposition 6.1.6.7] if $\mathcal{C}$ is pointed, then Nm_δ exists and is an equivalence.

We go one step further: suppose $\mathcal{C}$ is preadditive and f is 1-truncated, then if f is a relative finite groupoid, i.e. all fibers of f have finitely many connected components, such that each connected component is a classifying space of finite group, then δ is 0-truncated with finite fibers and by [[Lur09], Proposition 6.1.6.12], Nm_δ exists and is an equivalence.

Apply this machinery to $f : \mathrm{B}G \to *$ we have:

Definition 1.3 Let G be a finite field. Let $\mathcal{C}$ be a stable ∞-category with limits and colimits indexed by $\mathrm{B}G$. The Tate construction is an exact functor

$$\begin{aligned}(\cdot)^{tG} : \mathcal{C}^{\mathrm{B}G} &\to \mathcal{C}\\ X &\mapsto \mathrm{cofib}\left(X_{hG} \overset{Nm_G}{\to} X^{hG}\right).\end{aligned} \tag{1.4}$$

We can adapt this to a relative version: suppose H is a finite normal topological subgroup of G, then the natural projection $f : \mathrm{B}G \to \mathrm{B}(G/H)$ induces a Tate construction

$$(\cdot)^{tH} : \mathcal{C}^{\mathrm{B}G} \to \mathcal{C}^{\mathrm{B}(G/H)}. \tag{1.5}$$

This applies to our primary example $\mathbb{T}$ and its finite cyclic subgroup C_p.

How does this norm map coincide with the classical one? We can take a M a G-module and HM the associated Eilenberg-MacLane spectrum in Sp, then $\pi_i(HM^{tG}) \cong \hat{H}^{-i}(M; G)$.

Finally we record the multiplicative nature of the Tate construction.

Proposition 1.4 *[[NS18], Theorem I.3.1.] The Tate construction is lax symmetric monoidal, moreover, there is a unique up to contractability way of putting a lax symmetric monoidal structure on $(\cdot)^{tG}$ such that $(\cdot)^{hG} \Rightarrow (\cdot)^{tG}$ is symmetric monoidal.*

The proof of this proposition relies on the Verdier quotient of the ∞-categories. Recall that for $W \subset \mathcal{C}^{\Delta_1}$ a subset of maps in $\mathcal{C}$, let $\mathcal{D}$ be the full subcategory spanned by it. There is an ∞-category $\mathcal{C}/\mathcal{D}$ and a map $F : \mathcal{C} \to \mathcal{C}/\mathcal{D}$ making every map in W an equivalence in $\mathcal{C}/\mathcal{D}$.

Apply to the setting $\mathrm{Sp}^{BG}_{\mathrm{ind}} \subset \mathrm{Sp}^{BG}$, stably generated by $\bigoplus_g X$. This is a subcategory where $X^{tG} \simeq 0$. We apply our machinery. This factors through $(\cdot)^{tG} : \mathrm{Sp}^{BG}/\mathrm{Sp}^{BG}_{\mathrm{ind}} \to \mathrm{Sp}$ and everything is clear.

Example 1.5 The Conne's cyclic operator, introduced in the first talk, gives rise to a $\mathbb{T}$-action on THH, this is equivalent to the description $\mathrm{THH}(A) \simeq A^{\otimes S^1}$.

2 The ∞-category of cyclotomic spectra

Recall a lax equalizer of two functors $F, G : \mathcal{C} \to \mathcal{D}$ is the pullback of the following diagram in Cat_∞:

$$\begin{tikzcd}
\mathrm{LEq}(F,G) \arrow[r] \arrow[d] & \mathcal{D}^{\Delta_1} \arrow[d, "{(\mathrm{ev}_0, \mathrm{ev}_1)}"'] \\
\mathcal{C} \arrow[r, "{(F,G)}"] & \mathcal{D} \times \mathcal{D}
\end{tikzcd}$$

i.e., a pair (c, f) such that $c \in \mathcal{C}$ and $f : F(c) \to G(c)$ in $\mathcal{D}$.

Definition 2.1 The ∞-category of cyclotomic spectra CycSp is defined as a lax equalizer in between the product of forgetful functor and the Tate constructions. More precisely,

$$\mathrm{CycSp} := \mathrm{LEq}\left(\mathcal{C}^{\mathrm{B}\mathbb{T}} \rightrightarrows \prod_p \mathcal{C}^{\mathrm{B}\mathbb{T}} \right). \tag{2.1}$$

Therefore, a cyclotomic spectrum is a pair $\left(X, \left(\varphi_p\right)_p\right)$ for which X is a spectrum with $\mathbb{T}$-action and $\varphi_p : X \to X^{tC_p}$.

There is a description of maps in the lax equalizers:

Proposition 2.2 *[[NS18], Proposition II.1.5.] Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors. Let $X, Y \in \mathrm{LEq}(F,G)$, given as pairs (c_X, f_X) and (c_Y, f_Y), where $c_X, c_Y \in \mathcal{C}$ and $f_X : F(c_X) \to G(c_X)$, $f_Y : F(c_Y) \to G(c_Y)$. Then there is an equivalence of spaces*

$$\mathrm{Map}_{\mathrm{LEq}(F,G)}(X,Y) \simeq \mathrm{Eq}(\mathrm{Map}_{\mathcal{C}}(c_X, c_Y) \overset{(f_X)^*G}{\underset{(f_Y)^*F}{\rightrightarrows}} \mathrm{Map}_{\mathcal{D}}(F(c_X), G(c_Y))). \tag{2.2}$$

Example 2.3

1. The sphere spectrum $\mathbb{S}$, equipped with trivial $\mathbb{T}$-action, can be lifted into a cyclotomic spectrum. By pulling back along $r : \mathrm{B}C_p \to *$ there is a map $\mathbb{S} \to \mathbb{S}^{hC_p}$

(in Sp), composing with the natural map we have $\varphi_p : \mathbb{S} \to \mathbb{S}^{tC_p}$. To make it $\mathbb{T} \cong \mathbb{T}/C_p$-equivariant, we consider the composite

$$\mathbb{S} \to \mathbb{S}^{h\mathbb{T}} \simeq \left(\mathbb{S}^{hC_p}\right)^{h\mathbb{T}/C_p} \to \left(\mathbb{S}^{tC_p}\right)^{h\mathbb{T}/C_p} \tag{2.3}$$

lifting $\mathbb{S} \to \mathbb{S}^{tC_p}$ to $\mathrm{Sp}^{\mathrm{B}\mathbb{T}}$.

2. We will see later that there is a cyclotomic structure on $\mathrm{THH}(R)$ for each associative ring spectrum R.

Now we make a very general definition of the topological cyclic homology.

Definition 2.4 For X a cyclotomic spectra, the topological cyclic homology $\mathrm{TC}(X)$ of X is defined to be the mapping spectra $\mathrm{Map}_{\mathrm{CycSp}}(\mathbb{S}, X)$.

The previous explicit description of the mapping space yields a sequence

$$\mathrm{TC}(X) \to X^{h\mathbb{T}} \underset{\prod_p \varphi_p^{h\mathbb{T}}}{\overset{\mathrm{can}}{\rightrightarrows}} \prod_p \left(X^{tC_p}\right)^{h\mathbb{T}}. \tag{2.4}$$

Where

$$\mathrm{can} : X^{h\mathbb{T}} \simeq \left(X^{hC_p}\right)^{h\mathbb{T}/C_p} \simeq \left(X^{hC_p}\right)^{h\mathbb{T}} \to \left(X^{tC_p}\right)^{h\mathbb{T}}. \tag{2.5}$$

Note that under the equivalence $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{S}) \simeq \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathrm{Sp})$ the above sequence can be lifted to a sequence of spectra. Moreover, this exact sequence can be simplified further, suppose X is connected algebra. We will explain it in the next chapter.

3 Tate diagonal and TC

In this section, we write $T_p(R) := (R \otimes \ldots \otimes R)^{tC_p}$. We firstly prove a property about this functor T_p, sometimes also called topological Singer construction.

Proposition 3.1 *T_p is exact.*

Proof. We check T_p preserves direct sums and extensions.

$$\begin{aligned} T_p(X_1 \oplus X_2) &\simeq (X_1 \otimes \ldots \otimes X_1 \oplus X_2 \otimes \ldots \otimes X_2)^{tC_p} \\ &\simeq (X_1 \otimes \ldots \otimes X_2)^{tC_p} \oplus (X_2 \otimes \ldots \otimes X_2)^{tC_p} \oplus \left(\bigoplus_{[i_1, \ldots, i_p]} \left(\bigotimes_{(i_1, \ldots, i_p)} X_{i_1} \otimes \ldots X_{i_p} \right) \right)^{tC_p} \end{aligned} \tag{3.1}$$

where $[i_1, \ldots, i_p]$ is a representative system of C_p-orbits on $\{1, 2\}^p - \{(1, \ldots, 1), (2, \ldots, 2)\}$, which isomorphic to C_p. Therefore, the third term is an induced C_p-spectrum and the Tate construction vanishes.

Next for a fiber sequence

$$X_1 \to X_0 \to X_2 \tag{3.2}$$

we can define a filtration on $T_p(X_0)$ where all the middle terms are induced spectra and the Tate construction kills them, hence what remains is only $(X_1 \otimes \ldots \otimes X_1)^{tC_p}$ and $(X_2 \otimes \ldots \otimes X_2)^{tC_p}$ and the sequence stays exact. ■

Remark 3.2 Note that replace the Tate construction by homotopy fixed point functor we get a functor $F_p : \mathrm{Sp} \to \mathrm{Sp}$, this is however not always exact. For example, $H\mathbb{Z}$.

Next we define the Tate diagonal $\Delta_p : \mathrm{id}_{\mathrm{Sp}} \to T_p$, allowing us to endow a cyclotomic structure on $\mathrm{THH}(R)$ for $R \in \mathrm{Alg}$.

Definition 3.3 The Tate diagonal is the unique transformation $\Delta_p : \mathrm{id}_{\mathrm{Sp}} \to T_p$ associated to the map $\mathbb{S} \to \mathbb{S}^{hC_p} \to \mathbb{S}^{tC_p}$ under the equivalence $\mathrm{Fun}^{\mathrm{ex}}(\mathrm{id}, T_p) \simeq \mathrm{Fun}^{\mathrm{lex}}(\mathbb{S}, T_p(\mathbb{S}) \simeq \Omega^\infty \mathbb{S}^{tC_p})$.

Proposition 3.4 *The Tate diagonal* $\Delta_p : \mathrm{id}_{\mathrm{Sp}} \to T_p$ *can be uniquely symmetric monoidally refined.*

Now we use the Tate diagonal to equip a cyclotomic structure on THH. For each prime p, we may consider the following diagram.

$$\begin{tikzcd}
R \arrow[r, "\Delta_p"] \arrow[d, "i"'] & T_p(R) \arrow[r, "\psi^{tC_p}"] & \mathrm{THH}(R)^{tC_p} \\
\mathrm{THH}(R) \arrow[urr, dashed, "\varphi_p"'] & &
\end{tikzcd}$$

This works for any prime p, and therefore $\mathrm{THH}(R)$ is a cyclotomic spectrum.

Definition 3.5 Let $R \in \mathrm{Alg}$, the topological cyclic homology $\mathrm{TC}(R)$ of R is $\mathrm{TC}(\mathrm{THH}(R))$.

It's worth mentioning the negative topological cyclic homology $\mathrm{TC}^-(R) := \mathrm{THH}(R)^{h\mathbb{T}}$ and topological periodic homology $\mathrm{TP}(R) := \mathrm{THH}(R)^{t\mathbb{T}}$ here. The motivation for these definitions come from the computation for $\mathrm{HH}(R)$, R a discrete ring.

Conversely, we can also use (2.4) as a definition for topological cyclic homology of ring spectrum. Suppose R is connected, then ("combining all Frobenii") this is equivalent to the sequence

$$\mathrm{TC}(R) \to \mathrm{TC}^-(R) \underset{\varphi}{\overset{\mathrm{can}}{\rightrightarrows}} \mathrm{TP}(R)^\wedge \tag{3.3}$$

where

$$\varphi : (\mathrm{THH}(R))^{h\mathbb{T}} \to (\mathrm{THH}(R))^{t\mathbb{T}} \overset{\prod \varphi_p^{h\mathbb{T}}}{\to} \prod_p \left((\mathrm{THH}(R))^{tC_p}\right)^{h\mathbb{T}} \simeq \left((\mathrm{THH}(R))^{t\mathbb{T}}\right)^{\wedge} \tag{3.4}$$

The last completion denotes the profinite completion of all primes p. This needs us to understand the following proposition:

Lemma 3.6 *[[NS18], Lemma II.4.2] If X is a bounded below spectrum, then the canonical map*

$$X^{t\mathbb{T}} \to \left(X^{tC_p}\right)^{h\mathbb{T}} \tag{3.5}$$

is a p-completion, if X is p-completed, then this map is an equivalence.

4 Tate-valued Frobenius

We say some words about the Tate-valued Frobenius, and its relation to the cyclotomic Frobenius $\mathrm{THH}(R) \to (\mathrm{THH}(R))^{tC_p}$ for R an $\mathcal{E}_\infty$-ring. This is a refinement of the Frobenius map on commutative rings.

Now if $R \in \mathrm{CAlg}^{\mathrm{B}C_p}$, then the multiplication map $m : R \otimes \ldots \otimes R \to R$ naturally is C_p-equivariant, and induces a map $m^{tC_p} : (R \otimes \ldots \otimes R)^{tC_p} \to R^{tC_p}$. Compose with the T_p functor we have

$$R \overset{T_p}{\to} (R \otimes \ldots \otimes R)^{tC_p} \overset{m^{tC_p}}{\to} R^{tC_p} \tag{4.1}$$

of which we call Tate-valued Frobenius.

Example 4.1 For A a commutative ring and HA the associated Eilenberg-MacLane spectrum, the induced map of Tate-valued Frobenius $\varphi_p : HA \to HA^{tC_p}$ on π_0 recovers the usual Frobenius $\varphi : A \to A/p$, as $\pi_0 HA^{tC_p} = \hat{H}^0(A, C_p) \cong A/p$ by previous discussions.

In fact, this Frobenius map must factor through the Frobenius on THH due to the universal property of THH.

Theorem 4.2 *Let $\pi : \mathrm{THH}(R) \to R$ be the $\mathbb{T}$-equivariant map such that $\pi \circ i \simeq \mathrm{id}_R$ where $i : R \to \mathrm{THH}(R)$ the canonical map. Then the composition*

$$R \overset{i}{\to} \mathrm{THH}(R) \overset{\varphi_p}{\to} \mathrm{THH}(R)^{tC_p} \overset{\pi^{tC_p}}{\to} R^{tC_p} \tag{4.2}$$

is equivalent to the Tate-value Frobenius.

Proof. For the diagram

$$\begin{tikzcd}
& R \arrow[r, "i"] \arrow[d, "T_p"] & \mathrm{THH}(R) \arrow[d, "\varphi_p"] & \\
& (R \otimes \dots \otimes R)^{tC_p} \arrow[r, "\psi^{tC_p}"] & \mathrm{THH}(R)^{tC_p} \arrow[r, "\pi^{tC_p}"] & R^{tC_p}
\end{tikzcd}$$

If suffices to check the second row is multiplication map on $(R \otimes \dots \otimes R)$. Since this is an induced spectrum it suffices to check the $\pi \circ i$ is the identity, which is clear. ■

Bibliography

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[NS18] T. Nikolaus and P. Scholze, “On topological cyclic homology,” *Acta Mathematica*, vol. 221, no. 2, pp. 203–409, 2018, doi: 10.4310/ACTA.2018.v221.n2.a1.