Talk 10: Sheafification of intersection complexes

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1 Sheaf of intersection chains

Let X be a PL pseudomanifold, recall that if X has an admissible triangulation T, then every open set $U \subset X$ has an admissible triangulation S with respect to T, such that some subdivisions of S lie completely in T. So we may assume S lies in T.

Now for a PL i-chain $\xi \in C_i(X)$, define the restriction $\xi|_U$ be a PL chain on $C_i(U)$ as:

$$(\xi|_U)(\sigma') = \begin{cases} 0 & \text{if } \sigma' \text{ not contained in an } i\text{-simplex of } T \\ \xi(\sigma) & \text{if } \sigma' \text{ contained in a compatibly oriented } i\text{-simplex } \sigma \in T \end{cases} \tag{1.1}$$

This defines a restriction map $C_i(X) \to C_i(U)$. Moreover, one checks the structure of stratification is compatible with intersecting with opens, hence also a restriction map $IC_i^{\bar{p}}(X) \to IC_i^{\bar{p}}(U)$.

This assignment $U \mapsto IC_i^{\bar{p}}(U)$ gives us a presheaf. To see it is indeed a sheaf, we pick an open cover $\{U_\alpha\}$ of X and a family of i-chains $\{\xi_\alpha\}$ such that $\xi_\alpha|_{U_\alpha \cap U_\beta} = \xi_\beta|_{U_\alpha \cap U_\beta}$. We define a intersection chain $\xi \in IC_i(X)$ as

$$\xi(\sigma) = \xi_\alpha(\sigma_\alpha) \tag{1.2}$$

for any σ_α contained in $\sigma \cap U_\alpha$. This shows the unique gluing is fulfilled and $\underline{IC}_{\bar{p}}^{-i}(X)$ is a sheaf.

Definition 1.1 The sheaf-complex of perversity $\bar{p}$ intersection chains $\underline{IC}_{\bar{p}}^{\bullet}(X)$ is given in degree j by

$$\begin{gathered} \left(\underline{IC}_{\bar{p}}^{\bullet}(X)\right)^j = \underline{IC}_{\bar{p}}^{j}(X) = \text{Sheaf}\left(U \mapsto IC_{-j}^{\bar{p}}(U)\right) \\ \partial_j : \underline{IC}_{\bar{p}}^{j}(X) \to \underline{IC}_{\bar{p}}^{j+1}(X) \end{gathered} \tag{1.3}$$

Suppose $K \subset X$ is closed, we may choose a triangulation on K by a subcomplex of X. Sadly, although this makes the map $C_i(X) \to C_i(K)$ surjective, it cannot recover every intersection chain, as the intersection condition implies a chain never lies in the singular set and a naive lifting is impossible.

So if we want to show $\underline{IC}_{\bar{p}}^{\bullet}(X)$ is soft, we need to consult the following theorem:

Theorem 1.2 (Godement) If X is a paracompact space and $\underline{A}$ a sheaf on X, then for any $K \subset X$ closed we have

$$\Gamma(K, \underline{A}) = \varinjlim_{U \supset K} \Gamma(U, \underline{A}) \tag{1.4}$$

Theorem 1.3 $\underline{IC}^\bullet_{\bar{p}}(X)$ is soft.

Proof: By Godement's theorem for ant $K \subset X$ closed, a section $\xi \in \Gamma\big(K, \underline{IC}^{-i}_{\bar{p}}(X)\big)$ is represented by a section $\hat{\xi} \in \Gamma\big(U, \underline{IC}^{-i}_{\bar{p}}(X)\big)$ for $U \supset K$ open. Denote $K(T)$ be the set of vertices of an admissible triangulation T on U intersecting K.

Now we choose a triangulation T on U such that $\hat{\xi}$ is simplicial on X and every $U \cap X_i$ is covered by a subcomplex of T, and

$$N = \bigcup_{v \in K(T)} \mathrm{St}(v, T') \tag{1.5}$$

is closed in X, where T' the first bayercentric division of T and $\mathrm{St}(v, T')$ the closed star of v in T'.

From construction we know for each vertex $v \in T$, $\hat{\xi} \cap \mathrm{St}(v, T')$ is in $IC_i^{\bar{p}+\bar{0}}(X) = IC_i^{\bar{p}}(X)$ because $\mathrm{St}(v, T')$ is transverse to the strata of X. Therefore

$$\hat{\xi} \cap N \in IC_i^{\bar{p}}(X) = \Gamma\big(X; \underline{IC}^{-i}_{\bar{p}}(X)\big) \tag{1.6}$$

a global section restricts to $\xi \in \Gamma\big(K, \underline{IC}^{-i}_{\bar{p}}(X)\big)$. ■

We want to have an axiomatic characterization of $\underline{IC}^\bullet_{\bar{p}}(X)$ in the derived category $D^b(X)$. This makes it possible to compare with another constructions.

Definition 1.4 Let X be a topological space and F a decreasing filtration on it with $F^{-1}(X) = \emptyset$ and $F^n(X) = X$.

1. A complex of sheaves $\underline{A}^\bullet \in C(X)$ is called cohomologically locally constant with respect to F if for any i and $0 \le j \le n$, $\underline{H}^i(\underline{A}^\bullet)|_{X_j - X_{j-1}}$ is locally constant.
2. We say that $\underline{A}^\bullet$ is constructible with respect to F if $\underline{A}^\bullet$ is cohomologically locally constant with finitely generated stalks.

We denote the subcategory of constructible sheaves $D^b_c(X) \subset D^b(X)$.

Recall we write U_k for $X - X_{n-k}$ for deleted distinguished neighborhoods. Let $i_k : U_k \hookrightarrow U_{k+1}$ and $j_k : U_{k+1} - U_k \hookrightarrow U_k$ be inclusions.

From now on, all simplicial chains take the coefficients in $\mathbb{R}$.

Definition 1.5 A complex of sheaves $\underline{A}^\bullet \in D^b_c(X)$ satisfies the set of axioms [AX], if

(AX0): (Normalization) $\underline{A}^\bullet|_{X-\Sigma} \cong \mathbb{R}_{X-\Sigma}[n]$ the constant sheaf.

(AX1): (Lower bound) $\underline{H}^i(\underline{A}^\bullet) = 0$ for $i < -n$.

(AX2): (Vanishing) $\underline{H}^i\Big(\underline{A}^\bullet|_{U_{k+1}}\Big) = 0$ for $i > \bar{p}(k) - n, k \ge 2$.

(AX3): (Attaching) The maps

$$\underline{H}^i\Big(j_k^* \underline{A}^\bullet|_{U_{k+1}}\Big) \to \underline{H}^i\Big(j_k^* R i_{k*} i_k^* \underline{A}^\bullet|_{U_{k+1}}\Big) \tag{1.7}$$

are isomorphisms for all $i \leq \bar{p}(k) - n$ and $k \geq 2$.

Let's say something more about the meaning of these axioms. The normalization property (AX0) says away from singularities the intersection sheaf is just trivial. The vanishing property (AX2) tells us away from codimension k-stratum, the intersection sheaf has cohomological degree at most $\bar{p}(k)$, explaining the name "perversity".

(AX3) is the so called attaching property. The attaching map measures the difference of $\underline{A}^\bullet|_{U_{k+1}}$ and adjunction from U_k on the complement, in other words the possibility of reconstructing $\underline{A}|_{U_{k+1}}$ from pushing forward $\underline{A}|_{U_k}$. The property simply says(up to quasi-isomorphism) this is true for degrees smaller than perversity $\bar{p}$.

Remark 1.6

1. We could generally replace the constant sheaf $\underline{\mathbb{R}}_{X-\Sigma}$ in (AX0) with some locally constant system on $X - \Sigma$.
2. The constructibility is redundant, see Section 2.

Theorem 1.7 Let X^n be a oriented PL stratified pseudomanifold, then $\underline{IC}^\bullet_{\bar{p}}(X)$ satisfies [AX].

We will devote most of the time of this talk to proving this theorem.

Lemma 1.8 $\underline{IC}^\bullet_{\bar{p}}(X)$ satisfies (AX0).

Proof: Consider the sheafification of all $\mathbb{R}$-valued PL chains on opens of X, denoted by $\underline{C}^\bullet(X)$ and there's a canonical morphism

$$\underline{IC}^\bullet_{\bar{p}}(X) \to \underline{C}^\bullet(X) \tag{1.8}$$

induced by inclusion. On $X - \Sigma$ this turns out to be an isomorphism since chains on $X - \Sigma$ does not intersect Σ at all.

Passing to derived category, it suffices to show

$$\underline{H}^{-i}(\underline{C}^\bullet(X)|_{X-\Sigma}) = \begin{cases} \underline{\mathbb{R}}_{X-\Sigma} & \text{if } i = n \\ 0 & \text{if } i \neq n \end{cases} \tag{1.9}$$

Inspect the definition, for $U \subset X - \Sigma$ open we have

$$\begin{gathered} \underline{H}^{-i}(\underline{C}^\bullet(X)|_{X-\Sigma}) = \mathrm{Sheaf}(U \mapsto H^{-i}(\Gamma(U; \underline{C}^\bullet(X)))) \\ \mathrm{Sheaf}(U \mapsto H_i(U)) \end{gathered} \tag{1.10}$$

So if $i \neq n$, then for a small neighbourhood $U \cong \mathbb{R}^n$ and the Borel-Moore homology gives us $H_i(\mathbb{R}^n) = 0$. For $i = n$ by universal coefficient theorem we get

$$H_n(U) \cong \mathrm{Hom}(H^n_c(U), \mathbb{R}) \tag{1.11}$$

and $\underline{H}^{-n}(\underline{C}^\bullet(X)|_{X-\Sigma})$ is just the orientation sheaf of $X - \Sigma$. And as X is oriented, it is just $\underline{\mathbb{R}}_{X-\Sigma}$. ■

Proposition 1.9 Let $x \in X$ and U be a distinguished neighborhood of x in X. Then the restriction map induces an isomorphism:

$$IH_i^{\bar{p}}(U) \xrightarrow{\cong} \underline{H}^{-i}\big(\underline{IC}^\bullet_{\bar{p}}(X)\big)_x \tag{1.12}$$

Sketch of proof: We use the stratum preserving PL isomorphism for distinguished neighborhoods $(-1,+1)^{n-k} \times (L \times [0,1)/\sim) \cong U$ and construct a fundamental system of U as for $1 \geq \varepsilon > 0$ set

$$N_\varepsilon = (-\varepsilon, \varepsilon)^{n-k} \times (L \times [0, \varepsilon)/\sim) \tag{1.13}$$

From the calculation of previous talk we know for $\delta < \varepsilon$, the restriction $N_\delta \subset N_\varepsilon$ induces a quasi-isomorphism on intersection chains(as they are both quasi-isomorphic to $\tau_{\geq k-\bar{p}(k)-1} IC^{\bar{p}}_\bullet(L)[k-n-1]$). Since the restriction is functorial with respect to inclusions we have

$$IH_i^{\bar{p}}(U) \cong \lim_{\varepsilon \to 0} IH_i^{\bar{p}}(N_\varepsilon) \tag{1.14}$$

Now by the definition of stalks of cohomology sheaf the assertion is true. ■

The same story is true for deleted distinguished neighborhoods., just notice

$$IH_i^{\bar{p}}(L)[k-n-1] \cong IH_i^{\bar{p}}\big(N_\varepsilon - \big((-\varepsilon,\varepsilon)^{n-k} \times \{c\}\big)\big) = IH_i^{\bar{p}}(N_\varepsilon \cap U_k) \tag{1.15}$$

Proposition 1.10 Let $x \in X_{n-k} - X_{n-k-1}$ and U be a distinguished neighborhood of x. The restriction from $U \cap U_k$ induces an isomorphism:

$$IH_i^{\bar{p}}(U \cap U_k) \xrightarrow{\cong} \underline{H}^{-i}\Big(i_{k*}\underline{IC}^\bullet_{\bar{p}}(X)|_{U_k}\Big)_x \tag{1.16}$$

Lemma 1.11 $\underline{IC}^\bullet_{\bar{p}}(X)$ satisfies (AX2). i.e. $\underline{H}^j\Big(\underline{IC}^\bullet_{\bar{p}}(X)|_{U_{k+1}}\Big) = 0$ for $j > \bar{p}(k) - n, k \geq 2$.

Proof: We check it on stalks. Let $x \in X_{n-l} - X_{n-l-1}, l \leq k$ and U a distinguished neighborhood of x in X, then by Proposition 1.9:

$$\underline{H}^j\big(\underline{IC}^\bullet_{\bar{p}}(X)\big)_x \cong IH^{\bar{p}}_{-j}(U) \tag{1.17}$$

But we have already seen in previous talk how the intersection homology looks like for distinguished neighborhoods, which is

$$IH^{\bar{p}}_{-j}(U) \cong IH^{\bar{p}}_{-j}(\mathbb{R}^{n-l} \times c^\circ L) \cong \begin{cases} IH^{\bar{p}}_{-j-(n-l+1)}(L) & \text{if } -j \geq n - \bar{p}(l) \\ 0 & \text{if } -j < n - \bar{p}(l) \end{cases} \tag{1.18}$$

As $l \leq k$ and $\bar{p}(l) \leq \bar{p}(k)$, insert the condition $j > \bar{p}(k) - n$ we win. ■

Lemma 1.12 $\underline{IC}^{\bullet}_{\bar{p}}(X)$ satisfies (AX3), i.e. the adjunction

$$\underline{IC}^{\bullet}_{\bar{p}}(X)|_{U_{k+1}} \to i_{k*}i_k^*\left(\underline{IC}^{\bullet}_{\bar{p}}(X)|_{U_{k+1}}\right) \tag{1.19}$$

induces an isomorphism on derived sheaves

$$\underline{H}^j\left(j_k^*\left(\underline{IC}^{\bullet}_{\bar{p}}(X)|_{U_{k+1}}\right)\right) \to \underline{H}^j\left(j_k^* i_{k*} i_k^*\left(\underline{IC}^{\bullet}_{\bar{p}}(X)|_{U_{k+1}}\right)\right) \tag{1.20}$$

for all $j \leq \bar{p}(k) - n$ and $k \geq 2$.

Proof: By Theorem 1.3 we may replace Ri_{k*} simply by i_{k*}. Again by the calculation on distinguished neighborhoods we get

$$IH^{\bar{p}}_{-j}(U) \cong IH^{\bar{p}}_{-j-(n-k+1)}(L) \cong IH^{\bar{p}}_{-j}(U \cap U_k) \tag{1.21}$$

where U a distinguished neighborhood of $x \in U_{k+1} - U_k$. Whence the restriction

$$IH^{\bar{p}}_{-j}(U) \to IH^{\bar{p}}_{-j}(U \cap U_k) \tag{1.22}$$

is an isomorphism. The proof finishes by using Proposition 1.9 and Proposition 1.10 together with the following commutative diagram of restrictions:

$$\begin{array}{ccc}
IH^{\bar{p}}_{-j}(U) & \xrightarrow{\cong} & IH^{\bar{p}}_{-j}(U \cap U_k) \\
\cong \Big\downarrow \; (1.12) & & (1.16) \; \Big\downarrow \cong \\
\underline{H}^j\left(\underline{IC}^{\bullet}_{\bar{p}}(X)\right)_x & \longrightarrow & \underline{H}^j\left(i_{k*}i_k^*\underline{IC}^{\bullet}_{\bar{p}}(X)\right)_x
\end{array}$$

■

Proof of Theorem 1.7: $\underline{IC}^{\bullet}_{\bar{p}}(X)$ is by definition bounded and by Proposition 1.9 constructible. Moreover it's obvious that $\underline{H}^i\left(\underline{IC}^{\bullet}_{\bar{p}}(X)\right)$ is bounded below. Now by Lemma 1.8, Lemma 1.11 and Lemma 1.12 we win. ■

There's an alternative description of (AX3). Let $\underline{B}$ an injective sheaf on X and $K \subset X$ closed. $j : K \to X$ and $i : X - K \to X$ the inclusions, for any open subset $U \subset X$, as $\underline{B}|_U$ is flabby, there's a short exact sequence:

$$0 \to \Gamma_{U \cap K}(U; \underline{B}) \to \Gamma(U; \underline{B}) \to \Gamma(U - K; \underline{B}) \to 0 \tag{1.23}$$

Take the colimit yields

$$0 \to \left(j_* j^! \underline{B}\right)_x \to \underline{B}_x \to \left(i_* i^* \underline{B}\right)_x \to 0 \tag{1.24}$$

In derived category we have a distinguished triangle

$$j_* j^! \underline{B}^{\bullet} \to \underline{B}^{\bullet} \to Ri_* i^* \underline{B}^{\bullet} \xrightarrow{[1]} \tag{1.25}$$

This gives us a functorial description of the kernel of the adjunction

$$\underline{B} \to i_* i^* \underline{B} \tag{1.26}$$

Set $\underline{B}^\bullet = j_k^* \underline{A}^\bullet$, then the (AX3)

$$\underline{H}^i\left(j_k^* \underline{A}^\bullet|_{U_{k+1}}\right) \xrightarrow{\cong} \underline{H}^i\left(j_k^* R i_{k*} i_k^* \underline{A}^\bullet|_{U_{k+1}}\right) \tag{1.27}$$

for $i \le \bar{p}(k) - n$ is true iff

$$\underline{H}^i\left(j_k^! \underline{A}^\bullet\right) = 0, i \le \bar{p}(k) - n + 1 \tag{1.28}$$

If furthermore M^m is a manifold and $f_x : \{x\} \to M$ be the inclusion, then the identity

$$f_x^! \underline{B}^\bullet \cong f_x^* \underline{B}^\bullet[-m] \tag{1.29}$$

holds for cohomologically locally constant sheaf $\underline{B}^\bullet$. Let $j_x : \{x\} \to X$ be the inclusion. Factoring j_x as

$$\{x\} \xrightarrow{f_x} X_{n-k} - X_{n-k-1} \xrightarrow{j_k} X \tag{1.30}$$

and using (1.29) we see (1.28) is equivalent to

$$\underline{H}^i\left(j_x^! \underline{A}^\bullet\right) = 0, i \le \bar{p}(k) - k + 1, x \in X_{n-k} - X_{n-k-1} \tag{1.31}$$

As the complex $j_x^! \underline{A}^\bullet$ is often referred to the costalk of $\underline{A}^\bullet$ at x. (AX3) expresses a costalk vanishing condition.

2 Deligne's sheaf

So far we have defined intersection homology only for PL stratified pseudomanifold, but in a letter to Kazhdan and Lustzig, Deligne proposed a sheaf-complex in bounded, constructible derived category defined on every topological stratified pseudomanifold satisfying [AX] automatically(see [2]).

A big advantage is that it simplifies many proofs, for example the generalized Poincaré duality will be a direct consequence of Verdier duality.

Definition 2.1 Let X be any n-dimensional topological stratified pseudomanifold. Deligne's sheaf $\underline{S}^\bullet$ for perversity $\bar{p}$ is defined by the formula

$$\underline{S}^\bullet = \tau_{\le \bar{p}(n)-n} R i_{n*} \ldots \tau_{\le \bar{p}(3)-n} R i_{3*} \tau_{\le \bar{p}(2)-n} R i_{2*} \mathbb{R}_{X-\Sigma}[n] \tag{2.1}$$

where $i_k : U_k \hookrightarrow U_{k+1}$ as in last section and $X - \Sigma = U_2$ the top stratum and $\tau_{\le l}$ the truncation functor on complexes.

Theorem 2.2 If $\underline{A}^\bullet \in D^b(X)$ satisfies [AX], then $\underline{A}^\bullet \cong \underline{S}^\bullet$.

Proof: Induction on the codimension of strata k. If $k = 2$, then by (AX0) the normalization property this is true. Suppose we have constructed $\underline{A}^\bullet|_{U_k} \cong \underline{S}^\bullet|_{U_k}$, then the adjunction morphism

$$\underline{A}^\bullet|_{U_{k+1}} \to Ri_{k*}i_k^*\underline{A}^\bullet|_{U_{k+1}} \tag{2.2}$$

induces a morphism on sheaf-complexes

$$\tau_{\leq \bar{p}(k)-n}\underline{A}^\bullet|_{U_{k+1}} \to \tau_{\leq \bar{p}(k)-n}Ri_{k*}i_k^*\underline{A}^\bullet|_{U_{k+1}} \tag{2.3}$$

Now by (AX2) and the induction hypothesis we obtain

$$\begin{aligned} \underline{A}^\bullet|_{U_{k+1}} \cong \tau_{\leq \bar{p}(k)-n}\underline{A}^\bullet|_{U_{k+1}} \to \tau_{\leq \bar{p}(k)-n}Ri_{k*}i_k^*\underline{A}^\bullet|_{U_{k+1}} \\ \cong \tau_{\leq \bar{p}(k)-n}Ri_{k*}\underline{A}^\bullet|_{U_k} \\ \cong \tau_{\leq \bar{p}(k)-n}Ri_{k*}\underline{S}^\bullet|_{U_k} \\ = \underline{S}^\bullet|_{U_{k+1}} \end{aligned} \tag{2.4}$$

extending the isomorphism on U_k. It remains to show it is an isomorphism on $U_{k+1} - U_k$, of which we use the attaching condition (AX3) in derived category:

$$j_k^*\tau_{\leq \bar{p}(k)-n}\underline{A}^\bullet|_{U_{k+1}} \cong j_k^*\tau_{\leq \bar{p}(k)-n}Ri_{k*}i_k^*\underline{A}^\bullet|_{U_{k+1}} \tag{2.5}$$

Combine this with (2.4) we have $\underline{A}^\bullet|_{U_{k+1}} \cong \underline{S}^\bullet|_{U_{k+1}}$. ■

Remark 2.3 We see the condition of constructible sheaves is redundant, as by construction $\underline{S}^\bullet$ is automatically constructible and thus we don't need to show $\underline{IC}^\bullet_{\bar{p}}(X)$ defined in the last section is constructible.

We shall give a formal definition of intersection sheaf complex.

Definition 2.4 Let X be a PL stratified pseudomanifold, the complex of intersection chain sheaf with perversity $\bar{p}$ on X, denoted by $\underline{IC}^\bullet_{\bar{p}}(X)$, is a bounded above sheaf-complex satisfying [AX]. The intersection homology with perversity $\bar{p}$ are the hypercohomology of such a complex:

$$IH_i^{\bar{p}}(X) := \mathcal{H}^{-i}\left(X; \underline{IC}^\bullet_{\bar{p}}(X)\right) \tag{2.6}$$

If $\bar{p} \leq \bar{q}$, then by construction of Deligne's sheaf $\underline{S}^\bullet$, the canonical inclusion $\tau_{\leq a} \to \tau_{\leq b}$ induces a morphism $\underline{IC}^\bullet_{\bar{p}}(X) \to \underline{IC}^\bullet_{\bar{q}}(X)$.

Bibliography

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