Descent of $\mathcal{E}_\infty$-rings

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What we mean next as category is always an $(\infty, 1)$-category, where we identify the 1 -category with its nerve.

1 Motivation: Faithfully flat descent

In this section we consider the theory in 1-category. Grothendieck proposed a faithfully flat descent theorem of commutative rings. That is, if $A \to B$ is faithfully flat, then

$$\mathrm{Mod}_A \simeq \lim_{\leftarrow}\left(\mathrm{Mod}_B \rightrightarrows \mathrm{Mod}_{B\otimes_A B} \substack{\rightarrow\\\rightarrow\\\rightarrow} \mathrm{Mod}_{B\otimes_A B\otimes_A B}\right) \tag{1.1}$$

This theorem is proved using Barr-Beck theorem, by showing $F : \mathrm{Mod}_A \rightleftarrows \mathrm{Mod}_B : G$ is comonadic, which means there is a comonad $T = FG$ over $\mathrm{Fun}(\mathrm{Mod}_B, \mathrm{Mod}_B)$ equipped with composition monoidal structure such that $\mathrm{Mod}_A \simeq \mathrm{LMod}_T(\mathrm{Mod}_B^{\mathrm{op}})^{\mathrm{op}}$ given by $M \mapsto (M \otimes B, M \otimes B \to M \otimes B \otimes B)$.

2 Grothendieck topologies on CAlg

We introduced the fpqc and (finite) étale topology over CAlg.

Definition 2.1 Let $\varphi : A \to B$ be a morphism of $\mathcal{E}_\infty$-rings. φ is said to be (faithfully) flat if
1. $\pi_0 A \to \pi_0 B$ is (faithfully) flat and
2. φ induces an isomorphism of graded rings

$$\pi_0 B \otimes_{\pi_0 A} \pi_* A \to \pi_* B. \tag{2.1}$$

Remark 2.2 Similarly we can define étale morphism as morphism which is étale over π_0 and the map (2.1) is an equivalence.

This definition is essentially a generalization of faithfully flat maps of discrete rings. As [[Lur17], Corollary 7.2.1.22] we can show that if $A \to B$ is faithfully flat, then a morphism $M \to N$ of A-modules is an equivalence iff $M \otimes B \to N \otimes B$ is.

A collection of morphisms generate a finitary Grothendieck topology, and thus the category of sheaves as a coherent topos, if the following is hold:

Proposition 2.3 [[Lur18], Proposition A.3.2.1] *Let $\mathcal{C}$ be a category and S be a collection of morphisms in $\mathcal{C}$, suppose:*
1. *S contains all equivalences and is stable under composition.*
2. *$\mathcal{C}$ admits pullbacks and finite coproducts, and S is closed under them.*
3. *Finite coproducts in $\mathcal{C}$ are universal, i.e., given a diagram $\coprod_{1\le i\le n} A_i \to B \leftarrow B'$, the canonical map*

$$\coprod_{1\le i\le n}(A_i\times_B B')\to\left(\coprod_{1\le i\le n}A_i\right)\times_B B' \tag{2.2}$$

is an equivalence.

Then there is a finitary Grothendieck topology on $\mathcal{C}$ such that a sieve over C is a cover if and only if it contains a finite collection of morphism $\{C_i\to C\}_{1\le i\le n}$ such that $\coprod C_i\to C$ belongs to S.

Proposition 2.4 [[Lur18], Proposition B.6.1.3] *The collection of faithfully flat maps in* $\mathrm{CAlg}^{\mathrm{op}}$ *satisfies the assumptions in Proposition 2.3 and generates a Grothendieck topology, which we call the fpqc topology on* $\mathrm{CAlg}^{\mathrm{op}}$*.*

Remark 2.5

1. Similarly there is a "small" fpqc site over CAlg_R, the category of all R-algebras for R an $\mathcal{E}_\infty$-ring.
2. Both "big" and "small" fpqc sites are actually big, hence $\mathrm{Shv}_{\mathrm{fpqc}}(\mathrm{CAlg}_R^{\mathrm{op}})$ is not always an accessible localization of $\mathcal{P}(\mathrm{CAlg}_R^{\mathrm{op}}) := \mathrm{Fun}(\mathrm{CAlg}_R, \mathrm{An})$. Nevertheless, we will see a weaker version of descent like in classical setting in §4.

If we restrict our attention to the full subcategory $\mathrm{CAlg}_R^{\text{ét}}$, spanned by the étale R-algebras, then we have also a Grothendieck topology induced by faithfully flat maps, called the étale topology.

Remark 2.6 [[Lur17], Theorem 7.5.0.6] $\mathrm{CAlg}_{\pi_0 R}^{\text{ét}} \simeq \mathrm{CAlg}_R^{\text{ét}}$ for R an $\mathcal{E}_\infty$-ring. This makes the étale site easier to control than the fpqc site.

We can impose some finiteness condition on morphisms of $\mathcal{E}_\infty$-rings, which should be an extension of the theory on discrete rings.

Definition 2.7 Let $\varphi : A \to B$ be a flat morphism of $\mathcal{E}_\infty$-rings. We say φ is finite if φ exhibits $\pi_0 B$ as a finitely presented, of equivalently finitely generated projective module over $\pi_0 A$. We say φ is finite étale if it is both finite flat and étale.

Proposition 2.8 *There is a Grothendieck topology on $\left(\mathrm{CAlg}_R^{\text{ét}}\right)^{\mathrm{op}}$ generated by sieves of finite collection of morphisms $\{A\to A_i\}_{1\le i\le n}$ for which the induced map $A\to\prod_{1\le i\le n}A_i$ is finite flat and faithfully flat. We will refer to this topology as the finite étale topology on $\left(\mathrm{CAlg}_R^{\text{ét}}\right)^{\mathrm{op}}$.*

3 Universal descent and Barr-Beck-Lurie

We introduce pro objects, which is a generalization of presheaves.

Proposition 3.1 [[Lur09], Proposition 5.3.6.2] *Let $\mathcal{C}$ be a category, there is an category $\operatorname{Pro}(\mathcal{C})$ and a embedding $j : \mathcal{C} \to \operatorname{Pro}(\mathcal{C})$ with following universal properties:*
1. $\operatorname{Pro}(\mathcal{C})$ *has all small cofiltered limits.*
2. *Let $\mathcal{D}$ be a category with small cofiltered limits, let $\operatorname{Fun}'(\mathcal{C}, \mathcal{D})$ be those functors that preserve small cofiltered limits, then the embedding j induces an equivalence*

$$\operatorname{Fun}'(\operatorname{Pro}(\mathcal{C}), \mathcal{D}) \xrightarrow{\simeq} \operatorname{Fun}(\mathcal{C}, \mathcal{D}) \tag{3.1}$$

If $\mathcal{C}$ is accessible, we may identify $\operatorname{Pro}(\mathcal{C})$ with the full subcategory of $\operatorname{Fun}(\mathcal{C}, \mathrm{An})^{\mathrm{op}}$ spanned by functors that are left-exact and accessible.

Proposition 3.2 [[Lur09], Proposition 5.3.1.16] *Every pro-object $X \in \operatorname{Pro}(\mathcal{C})$ can be corepresented by a diagram $\mathcal{J} \to \mathcal{C}$ where $\mathcal{J}$ is a small cofiltered partially ordered set.*

The first step towards our wish is the ∞-categorical Barr-Beck theorem. It asks how we can recover the objects in the base category using an approximation via comonadic pair of functors.

Theorem 3.3 (Barr-Beck-Lurie) [[Lur17], Theorem 4.7.3.5] *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a pair of adjoint functors between categories. The adjunction is comonadic if and only if*
1. *F is conservative, i.e. preserves equivalences and*
2. *For every cosimplicial object $X^\bullet$ in $\mathcal{C}$ such that $F(X^\bullet)$ admits a splitting, $\operatorname{Tot}(X^\bullet)$ exists and*

$$F(\operatorname{Tot}(X^\bullet)) \simeq \operatorname{Tot}(F(X^\bullet)). \tag{3.2}$$

Example 3.4 Let $B \in \mathrm{CAlg}_A$, then the forgetful functor $\mathrm{Mod}_B \to \mathrm{Mod}_A$ is conservative and preserves limits and colimits, hence there is a right adjoint $\mathrm{Mod}_A \to \mathrm{Mod}_B$ and by Barr-Beck-Lurie this adjunction is comonadic.

However, we will need to consider the more general case. Given a comonadic adjunction as above, one can recover any object $C \in \mathcal{C}$ as the homotopy limit of the cobar construction

$$C \to \left(TC \rightrightarrows T^2C \substack{\rightarrow\\\rightarrow\\\rightarrow} \dots\right). \tag{3.3}$$

Here is an essential difference between 1-category and ∞-category. In 1-category, the homotopy limit of a cosimplicial diagram is a equalizer. But here it is infinite.

The following definition is universal among all morphisms satisfying this property. It was first discovered by homotopy theorist, when trying to understand modules over ring spectra in chromatic homotopy theory.

Definition 3.5 Let $\mathcal{C}$ be a stable category. A full subcategory $\mathcal{D} \subset \mathcal{C}$ is called think if $\mathcal{D}$ is closed under finite limits and colimits and under retracts. If $\mathcal{C}$ has a symmetric monoidal structure, then $\mathcal{D}$ is a think $\otimes$-ideal if it is in addition a $\otimes$-ideal.

Definition 3.6 A morphism $f : A \to B$ of $\mathcal{E}_\infty$-rings is called universal descent morphism if the thick $\otimes$-ideal generated by B is the whole Mod_A.

We can relate this definition to pro-objects and comonadicity.

Proposition 3.7 [[Mat16], Proposition 3.20] *Given $A \in \mathrm{CAlg}$, $A \to B$ admits descent if and only if the cosimplicial diagram $\mathrm{CB}^\bullet(B)$ defines a constant pro-object $\{\mathrm{Tot}_n \mathrm{CB}^\bullet(B)\}_{n\geq 0}$ which converges to A in $\mathrm{Pro}(\mathrm{Mod}_A)$, i.e., $\mathrm{CB}^\bullet_{\mathrm{aug}}(B)$ is a limit diagram.*

Proposition 3.8 [[Mat16], Proposition 3.22] *Given $A \in \mathrm{CAlg}$, if $A \to B$ admits descent then the adjunction $\mathrm{Mod}_A \rightleftarrows \mathrm{Mod}_B$ is comonadic, in particular, Mod_A can be covered from a total tower.*

The classical theorem of Grothendieck has the following $\mathcal{E}_\infty$-ring analogue.

Proposition 3.9 *Let $A \to B$ be a faithfully flat map of $\mathcal{E}_\infty$-rings such that $\pi_0(B)$ has a presentation as $\pi_0(A)$-algebra with at most $\aleph_k$ generators and relations for some $k \in \mathbb{N}$. Then $A \to B$ admits descent.*

This condition is essential. We may think it as the uncontrolled behaviour of fpqc An-valued sheaves: There could be some arbitrary big spaces! For a counterexample, see [Aok24] with a construction using boolean rings. For practical use, $\aleph_1$ is enough, just like that in condensed mathematics.

Finally we explain why this definition is the universal among all descent maps. This can be rephrased as a sheaf condition. We give some definition used first.

Definition 3.10 Let A be an $\mathcal{E}_\infty$-ring. A presentable (stable) category $\mathcal{C}$ is called A-linear if it is a module in the symmetric monoidal category $\mathrm{Mod}_{\mathrm{Mod}_A}(\mathrm{Pr}^L)$ of presentable categories over Mod_A.

We can think A-linear categories as a 2-categorical version of Mod_A.

Notation 3.11 Let $A \to B$ be a map of $\mathcal{E}_\infty$-rings and $\mathcal{C}$ be an A-linear category. We shall denote $\mathrm{Mod}_B(\mathcal{C})$ as the tensor product $\mathcal{C} \otimes_{\mathrm{Mod}_A} \mathrm{Mod}_B$ in Pr^L. Informally, $\mathrm{Mod}_B(\mathcal{C})$ is the target of an A-bilinear functor $\otimes_A : \mathcal{C} \times \mathrm{Mod}_B \to \mathrm{Mod}_B(\mathcal{C}), (X, M) \mapsto X \otimes_A M$.

Now we can state the main theorem.

Theorem 3.12 *Let A be an $\mathcal{E}_\infty$-ring and let $\mathcal{C}$ be a stable A-linear category. The construction $B \mapsto \mathrm{Mod}_B(\mathcal{C})$ determines a Pr^L-valued sheaf with respect to the universal descent topology on the category $\mathrm{CAlg}_A^{\mathrm{op}}$. Moreover, this is the finest topology such that this holds.*

4 Descent in fpqc topology

The universal descent topology is limited. We certainly want a (limited) descent theorem for finer topology, e.g. the fpqc topology. We have the following result:

Definition 4.1 Let $A \in \mathrm{CAlg}$ and $\mathcal{C}$ be a stable A-linear category. We say $\mathcal{C}$ satisfies flat (hyper)descent if the functor

$$\chi : \mathrm{CAlg}_A \to \widehat{\mathrm{Cat}_\infty}, B \mapsto B \otimes_A \mathcal{C} \tag{4.1}$$

is a (hypercomplete) fpqc sheaf.

Theorem 4.2 [[Lur18], Theorem D.6.3.1] *Let $A \in \mathrm{CAlg}_{\geq 0}$ and $\mathcal{C}$ be a Postnikov-complete prestable A-linear category, i.e. if $\mathcal{C} \simeq \lim_n \tau_{\leq n}\mathcal{C}$, then $\mathcal{C}$ satisfies flat hyperdescent.*

Proposition 4.3 [[Lur18], Corollary D.6.3.3] *Let A be an $\mathcal{E}_\infty$-ring, then the A-linear category Mod_A satisfies flat hyperdescent.*

Proof. by Example 3.3 we may assume $A = \mathbb{S}$, then A is connective. We claim it suffices to show $\mathrm{Mod}_A^{\mathrm{cn}}$ has flat hyperdescent, which is immediate from the last theorem. ■

Corollary 4.4 *The fpqc topology on $\mathrm{CAlg}^{\mathrm{op}}$ is subcanonical.*

5 More descent morphisms

Looking at the definition of descendable morphism, we notice it is only a pro-object-wise equivalence between the base ring and the totalization of the target. This equivalence may not preserve the ring structure in the intermediate step as finite totalizations, therefore, we can define a slightly different variant as in [AS25].

Definition 5.1 Let $f : R \to S$ be a morphism of $\mathcal{E}_\infty$-rings. f is said to be $\mathcal{E}_\infty$-descendable if the map of towers $\{R\} \to \{\mathrm{Tot}^n(S^{\wedge *+1})\}_n$ is a pro-equivalence in $\mathrm{Pro}(\mathrm{CAlg}(\mathrm{Mod}_R))$.

Proposition 5.2 [[AS25], Proposition 2.3] *Let $f : R \to S$ be a morphism of $\mathcal{E}_\infty$-rings. The followings are equivalent:*

1. *f is $\mathcal{E}_\infty$-descendable.*
2. *The map $R \to \mathrm{Tot}^n(S^{\wedge *+1})$ admits an $\mathcal{E}_\infty$-retraction for some $n \geq 0$.*
3. *If $\mathcal{C}$ is the smallest full subcategory of $\mathrm{CAlg}(\mathrm{Mod}_R)$ which contains the $\mathcal{E}_\infty$-algebra that admits a map from S and $\mathcal{C}$ is closed under finite limits and retractions, then $\mathcal{C}$ contains R.*

Example 5.3 KO $\to$ KU is descendable, but it is still unknown whether it is $\mathcal{E}_\infty$-descendable.

Bibliography

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