Relative-to-absolute descent

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1 Breuil-Kisin orientation

In this section we introduce the Breuil-Kisin orientation and show giving a Breuil-Kisin orientation is equivalent to giving a normal orientation on a prism (B, I). However, it is much more natural to give a Breuil-Kisin orientation instead of a normal one.

In order to achieve this, we need a canonical model of Breuil-Kisin twist, parallel to the stacky definition in [Dri20].

Definition 1.1 For (B, I) a prism, the Breuil-Kisin twist can be written as a limit

$$B\{1\} := \lim_{r} I_r / I_r^2 \tag{1.1}$$

where $I_r := I\varphi_B^*(I)...(\varphi_B^{r-1})^*(I)$ and the transition map is the unique map which after multiplying with p is the canonical map, provided (B, I) is transversal.

Remark 1.2 We have also the reduction to p compability, if (B, I) is a prism, then $B\{1\}/I_r \simeq I_r/I_r^2$, see [[BL22], Prop. 2.2.12].

Recall the set $\mathrm{Or}_R(M)$ of $R^\times$-torsors for an invertible R-module M. We will refer $\mathrm{Or}_B(I)$ to be the collection of usual orientations of (B, I) and $\mathrm{Or}_B(B\{1\})$ to the collection of Breuil-Kisin orientations.

Also notice the Breuil-Kisin twist comes with an induced Frobenius map from φ_B in the form of

$$\varphi_{B\{1\}} : B\{1\} \to B\{1\} \otimes_{B,\varphi} B \xrightarrow{\cong} I^{-1} \otimes_B B\{1\} \tag{1.2}$$

where the last isomorphism was constructed in [[BL22], Const. 2.2.14]. This map is in particular φ_B-semilinear. With this construction, we are able to express the compatibility of Breuil-Kisin orientation and the usual orientation of (B, I).

Proposition 1.3 *A Breuil-Kisin orientation $s : A \to A\{1\}$ determines a unique orientation d_s of (B, I) such that*

$$\varphi_{B\{1\}}(as) = d_s^{-1}\varphi_B(a)s. \tag{1.3}$$

The corresponding map

$$\mathrm{Or}_B(B\{1\}) \to \mathrm{Or}_B(I) \tag{1.4}$$

is equvariant with respect to the action $B^\times \to B^\times$ given by

$$u \mapsto \frac{u}{\varphi_B(u)}. \tag{1.5}$$

Proof. By [[BL22], Const. 2.2.14] the map

$$B\{1\} \otimes_{B,\varphi} B \to I^{-1} \otimes_B B\{1\}, s \otimes b \mapsto b\varphi_{B\{1\}}(s) \tag{1.6}$$

is an isomorphism, we see that $\varphi_{B\{1\}}$ is uniquely determined by the generator of $I^{-1} \otimes B\{1\}$, i.e. a generator of the form $d^{-1} \otimes s$. Now suppose $s' = us$ for $u \in B^\times$, then $d_{s'} = \frac{us}{\varphi_{B\{1\}}(us)} = \frac{u}{\varphi_B(u)} d_s$ by semilinearity. ∎

On the other hand, an orientation also determines a Breuil-Kisin orientation. In general however, this is not the inverse map of the above formula. The following special case helps though.

Proposition 1.4 *If $(B, (p))$ is crystalline, then there is a unique Breuil-Kisin orientation $s : B \to B\{1\}$ such that $d_s = p$ and $s = p^r \bmod I_r^2$ in $B\{1\}$.*

Proof. Specialize the existence of Breuil-Kisin orientation for q-de Rham prism as in [[BL22], Prop. 2.6.1] or chase the limit diagram defining Breuil-Kisin twist. ∎

Note $d_s = p$ is not enough to characterize s, as they might differ a unit u with $\frac{u}{\varphi(u)} = 1$ by Proposition 1.3.

We can immigrate the previous discussions to the setting of filtered prism $(\mathrm{F}^{\geq\star} A, I)$, assuming the filtration is complete. In particular, this works for our main example $(z^\star W(k)[[z]], E(z))$.

Lemma 1.5

1. *If $(F^{\geq\star} A, I)$ is a complete filtered crystalline prism, then (A, I) is canonically orientable in the above sense.*
2. *Conversely if (A, I) is filtered crystalline and complete, then for any $d \in I$ with image p in $\mathrm{gr}^0 I$, then there is a unique lift $s : A \to A\{1\}$ of the unique crystalline Breuil-Kisin orientation on $\mathrm{gr}^0 A$ such that $d_s = d$.*
3. *If $(F^{\geq\star} A, I)$ is a complete filtered crystalline prism, for any two $d, d' \in I$ with image p in $\mathrm{gr}^0 I$. Let $u \in (1 + F^{\geq 1} A)^\times$ with $d' = ud$. Let s and s' be the canonical filtered crystalline Breuil-Kisin orientation with $d_s = d$ and $d_{s'} = d'$, then*

$$s' = \left(\prod_{r \geq 0} \varphi^r(u)\right) s. \tag{1.7}$$

Proof. For 1, we may choose such an element $d \in I$ that reduced to $p \in \mathrm{gr}_F^0 I$ by surjectivity. This is an equation modulo $F^{\geq 1}$. Since $\delta(d)$ is a unit modulo $F^{\geq 1} A$ and $F^{\geq\star} A$ is complete, it is also a unit in A. This makes $(A, (d))$ a prism and $(A, (d)) \to (A, I)$ is a map of prisms. By the rigidity [[BS19], Prop. 3.5] $I = (d)$.

For 2, we notice each two lifts of Breuil-Kisin orientations differ by an element in $(1+F^{\geq 1}B)^{\times}$, since the map $u \mapsto \frac{u}{\varphi(u)}$ is an automorphism on $(1+F^{\geq 1}B)^{\times}$, there must exists a unique $u=1$, we conclude.

For 3, let $s' = vs$ and $d_{s'} = \frac{v}{\varphi(v)} d_s$. The equation $\frac{v}{\varphi(v)} = u$ is solved by $v = \prod_{r \geq 0} \varphi^r(u)$, by completeness this converges. ■

2 Relative-to-absolute descent

Now for (A, I) a prism we fix a Breuil-Kisin orientation $s : A \to A\{1\}$. The strategy explained before turned the relative syntomic complex into a fiber of Nyygaard-filtered prismatic complex

$$\mathbb{Z}_p(i)(R/A) \simeq \mathrm{fib}\Big(\mathrm{N}^{\geq i}\hat{\Delta}^{(1)}_{R/A}\{i\} \xrightarrow{\mathrm{can} - \varphi} \hat{\Delta}^{(1)}_{R/A}\{i\}\Big). \tag{2.1}$$

This is problematic because of the appearance of the twist. We explain how to use the Breuil-Kisin orientation s to turn this into an isomorphic complex

$$\mathrm{fib}\Big(\mathrm{N}^{\geq i}\hat{\Delta}^{(1)}_{R/A} \xrightarrow{\mathrm{can} - \varphi_i} \hat{\Delta}^{(1)}_{R/A}\Big). \tag{2.2}$$

Let $\varphi_A^*(-) \coloneqq - \hat{\otimes}_{A, \varphi_A} A$ be the (p, I)-completed scalar extension along φ_A, as by the construction we have $\varphi_A^*(A\{i\}) \cong I^{-i}A\{i\}$, hence for any $\overline{A}$-algebra R we have an A-linear map

$$\begin{aligned} \varphi_A^*\big(\Delta_{R/A}\{i\}\big) &\cong \varphi_A^*\big(\Delta_{R/A}\big) \otimes_A \varphi_A^*(A\{i\}) \\ &\xrightarrow{\varphi_{R/A} \otimes \mathrm{id}} \Delta_{R/A} \otimes_A I^{-i}A\{i\} \cong I^{-i}\Delta_{R/A}\{i\}. \end{aligned} \tag{2.3}$$

We will write $\Delta^{(1)}_{R/A}\{i\} \coloneqq \varphi_A^*\big(\Delta_{R/A}\{i\}\big)$. Fix a Breuil-Kisin orientation $s : A \to A\{1\}$, then there is an induced A-linear isomorphism $\Delta^{(1)}_{R/A} \xrightarrow{w(s^i)} \Delta^{(1)}_{R/A}\{i\}$.

For simplicity we may assume we are treating with Nyygaard-complete object.

Lemma 2.1 *The following diagram is commutative:*

$$\begin{tikzcd} \Delta^{(1)}_{R/A} \arrow[r, "\varphi_{i,R/A}"] \arrow[d, "w(s^i)"'] & I^{-i}\Delta_{R/A} \arrow[r, "w"] \arrow[d, "s^i"] & \Delta^{(1)}_{R/A} \arrow[d, "w(s^i)"'] \\ \Delta^{(1)}_{R/A}\{i\} \arrow[r, "\varphi_{R/A}"] \arrow[rr, bend right, "\varphi_{\Delta^{(1)}_{R/A}\{i\}}"'] & I^{-i}\Delta_{R/A}\{i\} \arrow[r, "w"] & \Delta^{(1)}_{R/A}\{i\} \end{tikzcd}$$

If $x \in N^{\geq i}\Delta^{(1)}_{R/A}$, then $\varphi_{i,R/A}(x) = \frac{\varphi_{R/A}}{d_s^i}(x)$. We will call the map $\varphi_{i,R/A}$ the i-th divided Frobenius. Henceforth, composing with w we have a map

$$\varphi_i : N^{\geq i}{\Delta}^{(1)}_{R/A} \to {\Delta}^{(1)}_{R/A}, x \mapsto \frac{\varphi(x)}{\varphi(d_s)^i} \tag{2.4}$$

which is φ-semilinear since w is.

Proof. Existence follows from the fact that both s^i and $w(s^i)$ are isomorphisms. We check the description of $\varphi_{i,R/A}$. As by Proposition 1.3 and the commutativity of the diagram, we have

$$\varphi_{R/A}(xw(s^i)) = d_s^{-i}\varphi_{R/A}(x)s^i = \varphi_{i,R/A}(x)s^i. \tag{2.5}$$

As s^i invertible, we see that $\varphi_{i,R/A}(x) = d_s^{-i}\varphi_{R/A}(x)$ in $I^{-i}{\Delta}_{R/A}$. By the definition of Nyygaard filtration, if $x \in N^{\geq i}{\Delta}^{(1)}_{R/A}$, we see that $\varphi_{i,R/A}(x) = d_s^{-i}\varphi_{R/A}(x) \in {\Delta}_{R/A}$. To see this, one possible way is to restrict ourself to the case where $L_{R/\overline{A}}$ has complete Tor-amplitude $[1,1]$, for example, if $R = \mathcal{O}_K/\varpi^n$, we can define the Nyygaard filtration by the pullback square

$$\begin{tikzcd}
N^{\geq j}{\Delta}_{R/A}\{i\} \arrow[r] \arrow[d] & I^{j-i}{\Delta}_{R/A}\{i\} \arrow[d] \\
{\Delta}^{(1)}_{R/A}\{i\} \arrow[r] & I^{-i}{\Delta}_{R/A}\{i\}
\end{tikzcd}$$

by [[BS19], Theorem 15.2]. ■

Definition 2.2 The oriented syntomic complex is

$$\mathbb{Z}_p(i)(R/A, s^i) := \mathrm{fib}\left(N^{\geq i}\hat{\Delta}^{(1)}_{R/A} \xrightarrow{\mathrm{can}\, - \varphi_i} \hat{\Delta}^{(1)}_{R/A}\right). \tag{2.6}$$

Theorem 2.3 *There is a natural equivalence*

$$\mathbb{Z}_p(i)(R/A) \simeq \mathbb{Z}_p(i)(R/A, s^i). \tag{2.7}$$

Now we switch to our main example, let $R = \mathcal{O}_K/\varpi^n$ with K a valuation field with k its perfect residue field. For an Eisenstein polynomial $E(z_0) \in W(k)[[z_0]]$, we may associate a prism structure on $W(k)[[z_0]]$. This extends to $W(k)[[z_0, ..., z_s]]$ and by prismatic envelop we have

$${\Delta}_{R/W(k)[[z_0,...,z_s]]} = W(k)[[z_0, ..., z_s]]\left\{\frac{z_0^n}{E(z_0)}, \frac{z_1 - z_0}{E(z_0)}, ..., \frac{z_s - z_0}{E(z_0)}\right\}^{\wedge} \tag{2.8}$$

where we send all z_i to ϖ. On this level, there is more compability than on the base prism, for example, we have a map from ${\Delta}_{R/W(k)[[z_0,...,z_s]]}$ to ${\Delta}_{R/W(k)[[z_0,...,z_{s+1}]]}$ taking z_0 to z_{s+1}. This will only be compatible with the Eisenstein polynomial $E(z_{s+1})$ as a prism structure, but in the envelop description, the element $\frac{z_{s+1}-z_0}{E(z_0)}$ ensures $E(z_{s+1}) - E(z_0)$ is divisible by $E(z_0)$, hence the choice $E(z_0)$ is not a matter.

By [[AKN23], Theorem 1.2(6)], (notice $W(k) \to W(k)[[z_0]]$ is flat and has Cech nerve $W(k)[[z_0, ..., z_s]]$ in degree s), we have the descent diagram

$$\mathbb{\Delta}_{R/W(k)}\{i\} \simeq \mathrm{Tot}\Big(\mathbb{\Delta}_{R/W(k)[[z_0]]}\{i\} \rightrightarrows \mathbb{\Delta}_{R/W(k)[[z_0,z_1]]}\{i\} \mathrel{\substack{\to\\\to\\\to}} ...\Big). \tag{2.9}$$

Notice if $R = \mathcal{O}_K/\varpi^n$, all terms are discrete and this is just a descent complex. Similarly, there is an F-filtered version. If $R = \mathcal{O}_K/\varpi^n$, then the cotangent complex $L_{R/A}$ statisfies conditions in [[AKN24], Theorem 2.20], and we may identify the Nyygaard completion with F-completion.

Definition 2.4 For a filtered commutative $z_0^\star W(k)[[z_0]]$-algebra R, we have an equivalence of diagrams

$$\hat{\mathbb{\Delta}}^{(1)}_{R/W(k)}\{i\} \simeq \mathrm{Tot}\Big(\hat{\mathbb{\Delta}}^{(1)}_{R/W(k)[[z_0]]}\{i\} \rightrightarrows \hat{\mathbb{\Delta}}^{(1)}_{R/W(k)[[z_0,z_1]]}\{i\} \mathrel{\substack{\to\\\to\\\to}} ...\Big) \tag{2.10}$$

referred as the F-completed relative to absolute descent diagram.

3 Structure maps

The descent complex is isomorphic to the cobar complex associated to a Hopf algebroid, as explained in [[AKN24], §4.3].

Definition 3.1 A Hopf algebroid is a pair of commutative rings (Γ_0, Γ_1) with following maps

$$\begin{tikzcd} \Gamma_0 \arrow[r, bend left, "\eta_L"] \arrow[r, bend right, "\eta_R"'] & \Gamma_1 \arrow[l, "\varepsilon"'] \arrow[loop below, "\iota"] \arrow[r, "\Delta"] & \Gamma_1 \otimes_{\eta_R, \Gamma_0, \eta_L} \Gamma_1 \end{tikzcd}$$

such that $(\mathrm{Spec}\ \Gamma_0, \mathrm{Spec}\ \Gamma_1)$ is a groupoid object in $\mathrm{CRing}^{\mathrm{op}}$. A comodule over a Hopf algebroid (Γ_0, Γ_1) is a right Γ_0-module M together with a coaction $M \to M \otimes_{\Gamma_0} \Gamma_1$ with identities regarding counit and coassociativity.

Note that if $\eta_L = \eta_R$, then the pair is also called commutative Hopf algebra, this is a very natrual object in algebraic topology, a prominent example is $(\pi_* E, E_*(E))$ for $E \in \mathrm{CAlg}$.

Construction 3.2 The pair $(W(k)[[z_0]], W(k)[[z_0, z_1]])$ is a commutative $W(k)$ Hopf algebroid, given by following assignments:

1. $\eta_L(z_0) = z_0$.
2. $\eta_R(z_0) = z_1$.
3. $\varepsilon(z_0) = z_0, \varepsilon(z_1) = z_0$.
4. $\iota(z_0) = z_1, \iota(z_1) = z_0$.

5. $\Delta(z_0) = z_0, \Delta(z_1) = z_2$ under the identification

$$W(k)[[z_0, z_1]] \otimes_{W(k)[[z_0]]} W(k)[[z_0, z_1]] \simeq W(k)[[z_0, z_1, z_2]]. \tag{3.1}$$

If R is a commutative $W(k)[[z_0]]$-algebra, then $((W(k)[[z_0]], R), (W(k)[[z_0, z_1]]), R)$ is a Hopf algebroid in the category of δ-pairs.

Theorem 3.3 *Suppose $L_{R/W(k)[[z_0]]}$ has p-completed Tor-amplitude $[0,1]$.*
1. The pair $\left(\mathbb{\Delta}_{R/W(k)[[z_0]]}, \mathbb{\Delta}_{R/W(k)[[z_0,z_1]]}\right)$ is completed filtered Hopf algebroid with respect to the Hodge-Tate filtration and the cobar resolution identifies with the Relative-to-absolute descent complex.
2. For each $i \in \mathbb{Z}$, $\mathbb{\Delta}_{R/W(k)[[z_0]]}\{i\}$ is a comodule over the Hopf algebroid $\left(\mathbb{\Delta}_{R/W(k)[[z_0]]}, \mathbb{\Delta}_{R/W(k)[[z_0,z_1]]}\right)$ and the descent complex identifies with its cobar complex.

Proof. Prismatic cohomology is a symmetric monomial functor over δ-pairs by [???], using the functoriality of Breuil-Kisin twists [[BL22], Prop. 2.5.1] we get an equivalence

$$\mathbb{\Delta}_{R/W(k)[[z_0,\dots,z_j]]}\{i\} \otimes_{\mathbb{\Delta}_{R/W(k)[[z_0]]}} \mathbb{\Delta}_{R/W(k)[[z_j,\dots,z_s]]} \to \mathbb{\Delta}_{R/W(k)[[z_0,\dots,z_s]]} \tag{3.2}$$

for any $s \geq 2, 0 \leq j \leq s$. Taking $j = 1$ and $s = 2$ produces the comultiplication map Δ, which is the only non-trivial map. ■

The above result also holds for F-completed terms and Nyygaard-filtered Frobenius-twisted terms, for the later one, remember the graded pieces are identified with conjugated-filtered pieces of Hodge-Tate filtration, which has symmetric monoidality inherited from the one of $\widehat{L\Omega}_{-/-}$.

Using the technique of prismatic envelope in [[AKN24], §3], we can describe the structure map $\mathsf{can} - \varphi_i$ in the case of $R = \mathcal{O}_K/\varpi^n$. By the previous discussion, we identify the computation of $\hat{\mathbb{\Delta}}_R\{i\}$ with the cobar complex associated to the Hopf algebroid $\left(\hat{\mathbb{\Delta}}_{R/W(k)[[z_0]]}, \hat{\mathbb{\Delta}}_{R/W(k)[[z_0,z_1]]}\right)$.

Proposition 3.4 *Let $a = \frac{z_0^n}{E(z_0)}$, $b = \frac{z_1 - z_0}{E(z_0)}$ and $u = \frac{E(z_1)}{E(z_0)} = 1 + \frac{E(z_1) - E(z_0)}{z_1 - z_0} b$. Then the structure maps in the Hopf algebroid $\left(\hat{\mathbb{\Delta}}_{R/W(k)[[z_0]]}, \hat{\mathbb{\Delta}}_{R/W(k)[[z_0,z_1]]}\right)$ are given by:*
1. $\eta_L : z_0 \mapsto z_0, a \mapsto a$.
2. $\eta_R : z_0 \mapsto z_1 = z_0 + E(z_0)b, a \mapsto \frac{z_1^n}{E(z_1)} = \frac{(z_0 + E(z_0)b)^n}{E(z_0)} u^{-1}$.
3. $\varepsilon : z_0 \mapsto z_0, a \mapsto a, b \mapsto 0$.
4. $\iota : z_0 \mapsto z_1, a \mapsto \frac{z_1^n}{E(z_1)}, b \mapsto \frac{z_0 - z_1}{E(z_1)} = -bu^{-1}$.
5. $\Delta : z_0 \mapsto z_0 \otimes 1, a \mapsto a \otimes 1, b \mapsto \frac{z_2 - z_0}{E(z_0)} = b \otimes 1 + u \otimes b$.

This works also for Frobenius-twisted case, where again use the prismatic envelope, we fix $f_0 = z_0^n \in N^{\geq 1}\hat{\mathbb{\Delta}}^{(1)}$ and $g_0 = z_1 - z_0$. Notice by [[AKN24], Proposition 3.33 &

3.34], the first one is sufficient to determine all generators, as they are free $W(k)$-module by some monomial iterating a, b under δ.

However, the later maps are not compatible with $\tilde{\delta} : N^{\geq\star}\hat{\Delta}^{(1)} \to N^{\geq p\star}\hat{\Delta}^{(1)}$ as they depend on orientations $E(z_0)$. This can be solved by calculating f_u, g_u under δ and passing to quotient, or we can use the connection, as introduced in the next section, we also prove the proposition by checking on graded pieces of F-filtration.

Finally we are also temped to describe the comodule structure on twists, note that by Proposition 1.4, $\hat{\Delta}_{R/W(k)[[z_0]]}\{i\}$ is the free $\hat{\Delta}_{R/W(k)[[z_0]]}$-module by a single element s^i such that $\varphi(s) = \frac{s}{E(z_0)}$, we have the following

Proposition 3.5 *The coaction*

$$\hat{\Delta}_{R/W(k)[[z_0]]}\{i\} \to \hat{\Delta}_{R/W(k)[[z_0,z_1]]}\{i\} \tag{3.3}$$

is given by sending $s^i \mapsto v^i s^i$ where

$$v = \prod_{r\geq 0} \varphi^r(u). \tag{3.4}$$

Proof. The coaction map is induced by $\eta_R : z_0 \mapsto z_1$ by definition, so it takes s with $d_s = E(z_0)$ to s' with $d_{s'} = E(z_1)$, by Lemma 1.5 (3), it takes s to vs. ∎

The above proposition is compatible with w, so it works for Frobenius-twisted version.

4 Connection

Even if we managed to know all structre maps, how can we compute this (likely) infinite descent complex? We invoke again the Hodge-Tate comparison theorem, and the fact that $L_{R/W(k)[[z_0]]}$ has Tor-amplitude $[0, 1]$, to see that $\hat{\Delta}_R$ and $\hat{\Delta}_R^{(1)}$ are cohomologically 1-dimensional. This means that, by good truncation we see the descent complex is quasi-isomorphic to the following 2-term complex

$$\hat{\Delta}^{(1)}_{R/W(k)[[z_0]]}\{i\} \to \ker\Big(\hat{\Delta}^{(1)}_{R/W(k)[[z_0,z_1]]}\{i\} \to \hat{\Delta}^{(1)}_{R/W(k)[[z_0,z_1,z_2]]}\{i\}\Big). \tag{4.1}$$

The second term looks still mysterious, and we investigate it now.

First we need some more structural results about Hopf algebroids, we fix $\Gamma = (\Gamma_0, \Gamma_1)$ a Hopf algebroid and M a Γ-right comodule, and we let Γ_1 to be a free left Γ_0-module (via η_L). For each left Γ_0-linear map $\theta : \Gamma_1 \to \Gamma_0$, we associate a natural map ∇_θ : $M \to M$ as the composite $M \to M \otimes_{\Gamma_0} \Gamma_1 \xrightarrow{\mathrm{id}\otimes\theta} M$.

Proposition 4.1 *The above construction is a set-theoretic isomorphism between left Γ_0-linear homomorphisms and natrual endomorphism of the forgetful functor from right Γ-comodule to abelian groups.*

Proof. We construct an inverse. For a natural endomorphism ∇ evaluating on Γ_1, we get a left Γ_0-module homomorphism $\nabla : \Gamma_1 \to \Gamma_1$ by naturality, since the left module structure on Γ_1 is given by composing the right coaction of Γ_0 and η_L. Composing with $\varepsilon : \Gamma_1 \to \Gamma_0$ we get a left linear homomorphism $\theta : \Gamma_1 \to \Gamma_0$.

The bijectivity follows from the following two commutative diagrams:

$$
\begin{tikzcd}
\Gamma_1 \arrow[dr] \arrow[drr, bend left, "\nabla_\theta"] \arrow[ddr, bend right, "\mathrm{id}"'] & & \\
& \Gamma_1 \otimes_{\Gamma_0} \Gamma_1 \arrow[r, "\mathrm{id} \otimes \theta"'] \arrow[d, "\varepsilon \otimes \mathrm{id}"] & \Gamma_1 \arrow[d, "\varepsilon"] \\
& \Gamma_1 \arrow[r, "\theta"] & \Gamma_0
\end{tikzcd}
$$

$$
\begin{tikzcd}
M \arrow[r, "\nabla"] \arrow[d] & M \arrow[d] \arrow[ddr, bend left, "\mathrm{id}"] & \\
M \otimes_{\Gamma_0} \Gamma_1 \arrow[r, "\mathrm{id} \otimes \nabla"] \arrow[drr, bend right, "\mathrm{id} \otimes \theta"'] & M \otimes_{\Gamma_0} \Gamma_1 \arrow[dr, "\mathrm{id} \otimes \varepsilon"] & \\
& & M
\end{tikzcd}
$$

Here, we can identify ∇ with $\mathrm{id}_M \otimes \nabla$ by naturality, since $- \otimes_{\Gamma_0} \Gamma_1$ is the colimit-preserving functor from right Γ_0-modules to right Γ-comodules. ∎

Under this identification, we expalin how to use ∇_θ to express some structure maps in Hopf algebroid.

If b_i is a basis of Γ_1 as a free left Γ_0-module, let $\theta^i : \Gamma_1 \to \Gamma_0$ be the dual basis of b_i and ∇_{θ^i} the associated natural transformation. For $x \in \Gamma_1$, we have

$$\Delta(x) = \sum_i \nabla_{\theta^i}(x) \otimes b_i \tag{4.2}$$

and for $x \in \Gamma_0$ we have

$$\eta_R(x) = \sum_x \nabla_{\theta^i}(x) b_i \tag{4.3}$$

where we view Γ_0 as a trivial right Γ-comodule.

Example 4.2 If Γ_1 is a free divided power algebra over Γ_0 with a generator y such that y is primitive, i.e. $\Delta(y) = y \otimes 1 + 1 \otimes y$. Let θ be the dual basis of y, then $\nabla_\theta(y^{[n]}) = y^{[n-1]} + \sum_{n \geq 2} \sum_i y^{[i]} \theta(y^{[n-2-i]})$ by the fact $\Delta(y^{[n]}) = \sum_i y^{[i]} \otimes y^{[n-i]}$ and the previous formula of Δ. In particular, $\varepsilon \circ \nabla_\theta^n$ is the dual basis of $y^{[n]}$. After suitable completion, every $\nabla(x)$ may be expressed as a power series on the chosen ∇_θ.

In this situation, we have a short exact sequence of left Γ_0-modules

$$0 \to \Gamma_0 \xrightarrow{\eta_L} \Gamma_1 \xrightarrow{\nabla_\theta} \Gamma_1 \to 0 \tag{4.4}$$

and after apply the functor $\mathrm{coBar}_\Gamma(M, -)$ we have a fiber sequence

$$\mathrm{coBar}_\Gamma(M, \Gamma_0) \to M \xrightarrow{\nabla_\theta} M \tag{4.5}$$

as the formular for ∇_θ can be written as a strictly upper triangular matrix with ones on the first off diagonal.

Now we specialize. Recall the prismatic envelope description gives us generators of $\hat{\Delta}_{R/W(k)[[z_0,z_1]]}$ as a free $\hat{\Delta}_{R/W(k)[[z_0]]}$-module, by $\prod \delta^u(b)^{e^u}$, where $e_u < p$. Similiarly, $\hat{\Delta}^{(1)}_{R/W(k)[[z_0,z_1]]}$ is freely generated by $\prod \tilde{\delta}^u(g)^{e_u}$ with $e_u < p$.

Set $F^{\geq\star}\hat{\Delta}^{\nabla}_{R/W(k)[[z_0]]}\{i\} := F^{\geq\star-1}\hat{\Delta}_{R/W(k)[[z_0]]}\{i-1\}$ and $F^{\geq\star}N^{\geq\star}\hat{\Delta}^{\nabla}_{R/W(k)[[z_0]]}\{i\} := F^{\geq\star-1}N^{\geq\star-1}\hat{\Delta}_{R/W(k)[[z_0]]}\{i-1\}$, we define the following map

$$\begin{aligned} \theta : F^{\geq\star}N^{\geq\star}\hat{\Delta}_{R/W(k)[[z_0,z_1]]}\{i\} &\to F^{\geq\star}N^{\geq\star}\hat{\Delta}^{\nabla}_{R/W(k)[[z_0]]}\{i\} \\ \theta : F^{\geq\star}\hat{\Delta}_{R/W(k)[[z_0,z_1]]}\{i\} &\to F^{\geq\star}\hat{\Delta}^{\nabla}_{R/W(k)[[z_0]]}\{i\} \end{aligned} \tag{4.6}$$

as the $\hat{\Delta}_{R/W(k)[[z_0]]}$-linear map taking bs^i (or $g_0 w(s^i)$) to s^{i-1} (or to $w(s^{i-1})$) and other monomials to 0. Notice here it is essential to take a F-completed prismatic cohomology, as we only define the maps over an F-complete basis.

Definition 4.3 The map $\nabla : F^{\geq\star}N^{\geq\star}\hat{\Delta}_{R/W(k)[[z_0]]}\{i\} \to F^{\geq\star}N^{\geq\star}\hat{\Delta}^{\nabla}_{R/W(k)[[z_0]]}\{i\}$ or $\nabla : F^{\geq\star}\hat{\Delta}_{R/W(k)[[z_0]]}\{i\} \to F^{\geq\star}\hat{\Delta}^{\nabla}_{R/W(k)[[z_0]]}\{i\}$ defined as the composition of $\eta_R - \eta_L$ and θ is called the connection map.

The main theorem of this section is the following:

Theorem 4.4 *There are commutative diagrams*

$$\begin{tikzcd} F^{\geq\star}\hat{\Delta}_{R/W(k)[[z_0]]}\{i\} \arrow[r, "\eta_R - \eta_L"] \arrow[d, "\mathrm{id}"] & F^{\geq\star}\hat{\Delta}_{R/W(k)[[z_0,z_1]]}\{i\} \arrow[r] \arrow[d, "\theta"] & \dots \\ F^{\geq\star}\hat{\Delta}_{R/W(k)[[z_0]]}\{i\} \arrow[r, "\nabla"] & F^{\geq\star}\hat{\Delta}^{\nabla}_{R/W(k)}[[z_0]]\{i\} \arrow[r] & 0 \end{tikzcd}$$

and

$$
\begin{tikzcd}
F^{\geq\star}N^{\geq\star}\hat{\Delta}^{(1)}_{R/W(k)[[z_0]]}\{i\} \arrow[r, "\eta_R-\eta_L"] \arrow[d, "\mathrm{id}"'] & F^{\geq\star}N^{\geq\star}\hat{\Delta}^{(1)}_{R/W(k)[[z_0,z_1]]}\{i\} \arrow[r] \arrow[d, "\theta"'] & \dots \\
F^{\geq\star}N^{\geq\star}\hat{\Delta}^{(1)}_{R/W(k)[[z_0]]}\{i\} \arrow[r, "\nabla"] & F^{\geq\star}N^{\geq\star}\hat{\Delta}^{(1),\nabla}_{R/W(k)}[[z_0]]\{i\} \arrow[r] & 0
\end{tikzcd}
$$

where the top rows are relative-to-absolute complexes and the vertical maps induce quasi-isomorphims between the rows.

Proof. The proof goes by first showing the existence of a divided power basis over $\left(\mathrm{gr}^\star_F\Delta_{R/W(k)[[z_0]]}, \mathrm{gr}^\star_F\Delta_{R/W(k)[[z_0,z_1]]}\right)$ and $\left(\mathrm{gr}^\star_F\hat{\Delta}^{(1)}_{R/W(k)[[z_0]]}, \mathrm{gr}^\star_F\hat{\Delta}^{(1)}_{R/W(k)[[z_0,z_1]]}\right)$, which is generated by primitive elements b and g_0, this makes it the situation as in Example 4.2.

Both elements are primitive as by Proposition 3.4. We first show g_0 is the generator of a divided power structure. Again as F-filtration is compatible with conjugate filtration, by crystalline comparison theorem, we have a divided power structure on the graded pieces of Frobenius-twisted term. On the other hand, explicitly we recall [[AKN24], Lemma 3.24] that

$$
g_u^p = \left(-p + d^{p^{u+1}}\lambda_u\right)g_{u+1} + d^{p^{u+1}}R'_u. \tag{4.7}
$$

Specialize to our cases, we see that

$$
R'_0 = \frac{\delta(z_1 - z_0)}{\delta(d)} = \frac{(z_0 + g_0)^p - z_0^p - g_0^p}{p\delta(d)} \tag{4.8}
$$

which is divisible by g_0 and has no constant terms as a g_0-polynomial. Inductively using the recursion formula for R'_u we see that all R'_us are constant term free, thus $d^{p^{u+1}-1}R'_u \in N^{\geq p^{u+1}}\hat{\Delta}^{(1)}_{R/W(k)[[z_0,z_1]]}$.

Passing to graded piece $\mathrm{gr}_F^{p^{u+1}}$ we have

$$
g_u^p = \left(-p + p^{p^{u+1}}\lambda_u\right)g_{u+1} + pd^{p^{u+1}-1}R'_u \tag{4.9}
$$

and thus g_u has a p-th divided power. Again inductively this shows that g_0 has all divided powers. Since $g_0^{[p^u]}$ differs with g_u by R'_u, i.e, by some polynomials in $g_0, \dots, g_{u-1}$, so they form a basis.

For the non-Frobenius-twisted case, we apply the relative divided Frobenius $\varphi_{i,R/A}$, and use the fact that we live in the F-completed world.

Next we show that with the map θ the sequence

$$
0 \to \Gamma_0 \overset{\eta_L}{\to} \Gamma_1 \overset{\nabla_\theta}{\to} \Gamma_1 \to 0 \tag{4.10}
$$

is F-completely exact both for Frobenius twisted and non-twisted Hopf algebroids. It suffices to check on graded pieces. In this case, we let $y = b$ or $y = g_0$, and the claim follows from Example 4.2.

Now we apply the functor $\mathrm{coBar}_\Gamma(M, -)$ with the trivial right comodule Γ_0 to the exact sequence, yields a map of complexes

$$\begin{tikzcd}
M \arrow[r] \arrow[d, "\mathrm{id}"'] & M \otimes_{\Gamma_0} \Gamma_1 \arrow[r] \arrow[d, "\mathrm{id} \otimes \theta"] & M \otimes_{\Gamma_0} \Gamma_1 \otimes_{\Gamma_0} \Gamma_1 \arrow[r] \arrow[d] & \dots \\
M \arrow[r, "\nabla_\theta"] & M \arrow[r] & 0
\end{tikzcd}$$

natural in M. If M is of the form $N \otimes_{\Gamma_0} \Gamma_1$ for a flat right Γ_0-module, then this becomes a quasi-isomorphim by exactness and the fact $- \otimes_{\Gamma_0} \Gamma_1$ is colimit-preserving, which is exactly our case. ∎

In particular, this theorem together with the analysis at the beginning of the section, gives us an isomorphism

$$\ker\left(F^{\geq\star}\hat{\Delta}_{R/W(k)[[z_0,z_1]]}\{i\} \to F^{\geq\star}\hat{\Delta}_{R/W(k)[[z_0,z_1,z_2]]}\{i\}\right) \xrightarrow{\theta} F^{\geq\star}\hat{\Delta}^{\nabla}_{R/W(k)[[z_0]]}\{i\} \tag{4.11}$$

We denote $\mathrm{can}^\nabla, \varphi^\nabla$ to be the maps after transporting can, φ along the isomorphism θ.

Corollary 4.5 *The syntomic complex $\mathbb{Z}_p(i)(R)$ is identified with the total fiber of the square*

$$\begin{tikzcd}
F^{\geq\star}N^{\geq i}\hat{\Delta}^{(1)}_{R/W(k)[[z_0]]} \arrow[r, "N^{\geq i}\nabla"] \arrow[d, "\mathrm{can} - \varphi_i"'] & F^{\geq\star}N^{\geq i}\hat{\Delta}^{(1),\nabla}_{R/W(k)[[z_0]]} \arrow[d, "\mathrm{can}^\nabla - \varphi_i^\nabla"] \\
F^{\geq\star}\hat{\Delta}^{(1)}_{R/W(k)[[z_0]]} \arrow[r, "\nabla"] & F^{\geq\star}\hat{\Delta}^{(1),\nabla}_{R/W(k)[[z_0]]}
\end{tikzcd}$$

All maps appear here are never linear, and are not compatible with any module structure on terms. In fact, switching z_0 and z_1 yields completely different θ and ∇! Moreover, we notice that the map φ_i^∇ is also uniquely characterized by the above square, therefore we don't need to calculate K and θ explicitly.

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