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Slice, décale and realize, motivically

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Abstract

We study the slice filtrations of motivic ring spectra and spectral sequences associated to them. Using décalage, we could relate Adams-Novikov spectral sequences with slice spectral sequences after suitable Betti realization and étale realization, this works in both characteristic zero and p cases.

Zusammenfassung

Wir betrachten die Slice-Filtrationen motivischer Ringspektren und die zugehörigen Spektralsequenzen. Mit dem sogenannten Décalage-Funktor können wir Isomorphismen zwischen der Spektralsequenz von Adams-Novikov und der aus der Slice-Filtration stammenden Spektralsequenz nach geeigneter Betti- und étale Realisierung konstruieren. Der Ansatz ist sowohl in Charakteristik 0 als auch in Charakteristik p anwendbar.

I imagined it infinite, made not only of eight-sided pavilions and of twisting paths

but also of rivers, provinces and kingdoms…

I though of a maze of mazes, of a sinuous, ever growing maze

which would take in both past and future and would somehow involve the stars.

Jorge Luis Borges, *The Garden of Forking Paths*

Contents

1 Introduction

Where there is a spectral sequence, there is a filtration. In this thesis, we will have a close inspection of this "common sense". In particular, the homotopy information of graded pieces of every filtration turns out to be a reasonable approximation of the homotopy information of the object, this approximation is realized via spectral sequences.

We will focus on the so called motivic ring spectra in motivic homotopy theory. In [Voe02] Voevodsky defined the slice filtration on these ring spectra and proposed a series of conjectures about it. several conjectures are related to the convergence of spectral sequences associated to certain slices, for example, the slice filtration on algebraic cobordism spectrum MGL.

Nowadays we know the behavior of the slice spectral sequence of MGL is more or less "topological": as discovered in [Lev15], under Betti realization it is isomorphic to the Adams-Novikov spectral sequence for calculating stable homotopy group of spheres. On the other hand, the slice filtration on MGL works as a "bridge" between stable motivic stems and topological stable stems as shown in Lemma 5.2.3 later. The goal of this thesis is to study these connections in a systematic way.

One of our main tools is called décalage, first mentioned in [Del71], and was explicitly used in the form of cosimplicial spectra by [Lev15], this tool is used to construct spectral sequences by translating higher pages into E_1-page. Recently [Ant24] gives a nice reinterpretation of the theory in the setting of stable ∞-categories, making it possible to extend old results into the ∞ -categorical setting. We introduce these results now.

Let k be an algebraically closed field of character zero. The embedding $k \hookrightarrow \mathbb{C}$ induces a Betti realization functor $\mathrm{Re}_{\mathbb{C}} : \mathcal{SH}(k) \to \mathcal{SH}$. Let $\mathbb{S}_k$ be the motivic sphere spectrum, i.e. the unit in $\mathcal{SH}(k)$ and let $\mathbb{S}_k^*$ be the slice filtration on it.

We give new proofs of following theorems:

Theorem 1 [[Lev15], Theorem 1] Consider the Adams-Novikov spectral sequence

$$E_2^{s,t}(AN) = \mathrm{Ext}^{s,t}_{\mathrm{MU}_*(\mathrm{MU})}(\mathrm{MU}_*, \mathrm{MU}_*) \Longrightarrow \pi_{t-s}\mathbb{S} \tag{1.1}$$

and the motivic Atiyah-Hirzebruch spectral sequence

$$E_1^{p,q}(AH) = \pi_{-p-q,0}(\mathrm{gr}^{-q}\mathbb{S}_k^*)(k) \Longrightarrow \pi_{-p-q,0}(\mathbb{S}_k)(k). \tag{1.2}$$

Then there is an isomorphism

$$\gamma_1^{p,q} : E_1^{p,q}(AH) \cong E_2^{3p+q,2p}(AN) \tag{1.3}$$

which induces a sequence of isomorphisms of complexes for $r \geq 1$

$$\oplus_{p,q}\, \gamma_r^{p,q} : \left(\oplus_{p,q} E_r^{p,q}(AH), d_r\right) \to \left(\oplus_{p,q} E_{2r+1}^{3p+q,2p}(AN), d_{2r+1}\right). \tag{1.4}$$

In other words, the Betti realization of the slice filtration spectral sequence in $\mathcal{SH}(k)$ is isomorphic to the classical Adams-Novikov spectral sequence up to indices.

Theorem 2 [[Lev15], Theorem 2] Fix a prime ℓ and the associated Brown-Peterson spectrum $\mathrm{BP}^{(\ell)}$. The isomorphism in Theorem 1 extends to an isomorphism of the ℓ-local Adams-Novikov spectral sequence

$$E_2^{s,t}(AN)_\ell = \mathrm{Ext}^{s,t}_{\mathrm{BP}^{(\ell)}_*(\mathrm{BP}^{(\ell)})}\left(\mathrm{BP}^{(\ell)}_*, \mathrm{BP}^{(\ell)}_*\right) \Longrightarrow \pi_{t-s}\mathbb{S} \otimes \mathbb{Z}_{(\ell)} \tag{1.5}$$

and the ℓ-local motivic Atiyah-Hirzebruch spectral sequence

$$E_1^{p,q}(AH)_\ell = \pi_{-p-q,0}(\mathrm{gr}^{-q}\mathbb{S}_k^\star)(k) \otimes \mathbb{Z}_{(\ell)} \Longrightarrow \pi_{-p-q,0}(\mathbb{S}_k)(k) \otimes \mathbb{Z}_{(\ell)}. \tag{1.6}$$

And we have our new result, answering the question in [[Lev15], Remarks 2.3]:

Theorem 3 Let k be any algebraically closed field of characteristic p. Let $\ell \neq p$ be a prime number. Let $\mathrm{MU}^\wedge_\ell$ be the ℓ-adic completion of complex cobordism. For simplicity We may denote $\mathbb{S}_k$ and MU to be their Bott inverted counterparts as explained in §5.3. The Adams-Novikov spectral sequence

$$E_2^{p,q}(AN)^\wedge_\ell = \mathrm{Ext}^{s,t}_{\mathrm{MU}^\wedge_{\ell,*}(\mathrm{MU}^\wedge_\ell)}\left(\mathrm{MU}^\wedge_{\ell,*}, \mathrm{MU}^\wedge_{\ell,*}\right) \Longrightarrow (\pi_{t-s}\mathbb{S})^\wedge_\ell \tag{1.7}$$

converges and is isomorphic to the ℓ-complete motivic Atiyah-Hirzebruch spectral sequence

$$E_1^{p,q}(AH)^\wedge_\ell = \pi_{-p-q,0}(\mathrm{gr}^{-q}\mathbb{S}_k^*[1\ /\ p])(k) \otimes \mathbb{Z}_\ell \Longrightarrow \pi_{-p-q,0}(\mathbb{S}_k[1\ /\ p])(k) \otimes \mathbb{Z}_\ell \tag{1.8}$$

with isomorphisms induced by the étale realization functor.

Outline of the thesis: In Chapter 2 we introduce the abstract theory of décalage as developed in [Ant24]. In Chapter 3 we review the basic motivic homotopy theory, putting an emphasize on the construction of algebraic cobordism as a Thom spectrum. In Chapter 4 we construct slice filtrations of motivic spectra and realization functors, proving exactness condition and descent properties of them. Putting all these ingredients together, we give proofs of our main results in Chapter 5. In Appendix we collect some tools from higher category theory and higher algebra used in this thesis.

Convention: What we mean of an ∞-category is always an $(\infty, 1)$-category in the sense of Lurie. We always identify an 1-category with its nerve. Every ring in this thesis is unitial.

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2 t-Structure on filtrations

2.1 Filtrations and chain complexes

Let $\mathcal{C}$ be an ∞-category with cofibers, an initial object 0 and a final object $*$.
Recall we can view $\mathbb{Z}$ as a category whose objects are the integers $n \in \mathbb{Z}$ and there is at most one map $n \to m$, which exists when $n \leq m$.

Definition 2.1.1 The ∞-category of decreasing filtrations $\mathrm{Fil}(\mathcal{C})$ is defined by

$$\mathrm{Fil}(\mathcal{C}) := \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathcal{C}) \tag{2.1.1}$$

Similarly, one can define the ∞-category of increasing filtrations by replacing $\mathbb{Z}^{\mathrm{op}}$ with $\mathbb{Z}$.
Intuitively, a filtration $\mathrm{F}^\star$ indexed by $\mathbb{Z}^{\mathrm{op}}$ may looks like a sequence

$$... \to \mathrm{F}^{s+1} \to \mathrm{F}^s \to \mathrm{F}^{s-1} \to ... \tag{2.1.2}$$

This definition makes $\mathcal{C}$ a full subcategory of $\mathrm{Fil}(\mathcal{C})$ by viewing each object $X \in \mathcal{C}$ as a constant filtration.

Definition 2.1.2 We will also consider a filtration on an object X of $\mathcal{C}$: it is a filtration $\mathrm{F}^\star$ together with a map $\mathrm{F}^\star \to X$, where X is viewed as a constant filtration. We denote the ∞-category of filtrations on objects as $\mathrm{Fil}(\mathcal{C})/\mathcal{C}$.

Consider the category $\mathbb{Z}^{\mathrm{op}}_+$ an extension of $\mathbb{Z}^{\mathrm{op}}$ with a new terminal object $-\infty$. We have an equivalence of ∞-categories:

$$\mathrm{Fil}(\mathcal{C})/\mathcal{C} \simeq \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}_+, \mathcal{C}). \tag{2.1.3}$$

Definition 2.1.3 A filtration $\mathrm{F}^\star$ on X is exhaustive if $X \simeq \mathrm{colim}_s \mathrm{F}^s =: \mathrm{F}^{-\infty}$. A filtration $\mathrm{F}^\star$ is complete if $\mathrm{F}^\infty := \lim_s \mathrm{F}^s \simeq 0$. By construction, all complete filtrations form a full subcategory of filtrations, to which we denote $\mathrm{Fil}_c(\mathcal{C})$.

Remark 2.1.4 Assuming $\mathcal{C}$ has sequential colimits, which ensures the existence of the realization $|\mathrm{F}^\star| := \mathrm{colim}_s \mathrm{F}^s$ in $\mathcal{C}$, each filtration $\mathrm{F}^\star$ can be viewed as a filtration on $|\mathrm{F}^\star| = \mathrm{F}^{-\infty}$. If $\mathrm{F}^\star$ is a filtration on X, then we have a canonical map $|\mathrm{F}^\star| \to X$ by taking the colimit, which is an equivalence iff $\mathrm{F}^\star$ is exhaustive.
It's also not hard to see the functor $|-| : \mathrm{Fil}(\mathcal{C}) \to \mathcal{C}$ is a left adjoint of the constant functor $\mathcal{C} \to \mathrm{Fil}(\mathcal{C})$. The picture we can bear in mind for this example is

$$\begin{tikzcd}
... \arrow[r] & \mathrm{F}^{s+1} \arrow[r] \arrow[dr] & \mathrm{F}^s \arrow[r] \arrow[d] & \mathrm{F}^{s-1} \arrow[r] \arrow[dl] & ... \\
& & \mathrm{F}^{-\infty} & &
\end{tikzcd}$$

Definition 2.1.5 Let $\mathrm{F}^\star$ be a filtration. The graded piece of $\mathrm{F}^\star$ of degree $s \in \mathbb{Z}$ is

$$\mathrm{gr}^s \mathrm{F}^\star := \mathrm{cofib}(\mathrm{F}^{s+1} \to \mathrm{F}^s) \tag{2.1.4}$$

More generally, for $i \le j$, let $\mathrm{gr}^{[i,j)}\mathrm{F}^\star = \mathrm{cofib}(\mathrm{F}^j \to \mathrm{F}^i)$. By definition we have $\mathrm{gr}^{[i,i)}\mathrm{F}^\star \simeq 0$ and $\mathrm{gr}^{[i,i+1)}\mathrm{F}^\star \simeq \mathrm{gr}^i\mathrm{F}^\star$. It's convenient to denote $\mathrm{gr}^{(-\infty,j)}\mathrm{F}^\star$ for $\mathrm{colim}_i \mathrm{gr}^{[i,j)}\mathrm{F}^\star$ and $\mathrm{gr}^{[i,\infty)}\mathrm{F}^\star$ for $\lim_j \mathrm{gr}^{[i,j)}\mathrm{F}^\star$.
If the context is clear, we may drop out $\mathrm{F}^\star$ and simply write $\mathrm{gr}^{[i,j)}$.

The graded piece $\mathrm{gr}^{[i,j)}$ itself has a filtration

$$\dots \to 0 \to \mathrm{gr}^{j-1} \to \mathrm{gr}^{[j-2,j)} \to \dots \to \mathrm{gr}^{[i,j)} \xrightarrow{\mathrm{id}} \mathrm{gr}^{[i,j)} \xrightarrow{\mathrm{id}} \dots \tag{2.1.5}$$

by the nature of cofiber sequences with gr^{j-1} put in the filtration degree $j-1$. The graded piece of $\mathrm{gr}^{[i,j)}$ would be

$$\mathrm{gr}^k\big(\mathrm{gr}^{[i,j)}\big) := \begin{cases} \mathrm{gr}^k(\mathrm{F}^\star) & \text{if } i \le k < j \\ 0 & \text{else} \end{cases} \tag{2.1.6}$$

Example 2.1.6 We can equip a cochain complex $X^\bullet \in \mathrm{Ch}^\star(\mathbb{Z})$ with the so-called truncation filtration $\sigma^\star X^\bullet$ by $\sigma^s X^\bullet = X^{\ge s}$. It's a complete exhaustive filtration by definition. Looking at the underlying homotopy type produces a filtration $\sigma^\star X$ on X, the image of $X^\bullet$ in $\mathrm{D}(\mathbb{Z})$ as derived ∞-category. The s-th graded piece of $\sigma^\star X^\bullet$ is $\mathrm{gr}^s(\sigma^\star X^\bullet) \cong X^s$ placed at the cohomological degree s, while the s-th graded piece of $\sigma^\star X$ is $\mathrm{gr}^s(\sigma^\star X) \cong X^s[-s]$.

Example 2.1.7 Let $\mathcal{SH}$ be the ∞-category of spectra with the usual Postnikov t-structure. Then for each $X \in \mathcal{SH}$ the Whitehead tower defines a filtration on X:

$$\dots \to \tau_{\ge n+1}X \to \tau_{\ge n}X \to \tau_{\ge n-1}X \to \dots \tag{2.1.7}$$

This is a complete and exhaustive filtration on X as the Postnikov t-structure is compatible with filtered colimits. The graded piece is $\mathrm{gr}^s X \simeq \pi_s X[s]$.

A common approach of understanding new objects in mathematics is finding a suitable filtration, taking the graded pieces, and trying to extract the information of the original object from them. One method of computing these information is via spectral sequences. For the remaining part of this chapter, we will introduce two methods for constructing spectral sequences associated to filtrations.

Definition 2.1.8 If $\mathcal{C}$ admits sequential limits, the inclusion $\mathrm{Fil}_c(\mathcal{C}) \subset \mathrm{Fil}(\mathcal{C})$ has a left adjoint, which is the completion of a filtration. Explicitly the filtration is given by

$$\widehat{\mathrm{F}^s} := \mathrm{cofib}(\mathrm{F}^\infty \to \mathrm{F}^s) \tag{2.1.8}$$

Suppose $\mathrm{F}^\star$ is a filtration on X, we set $\hat{X} := \mathrm{cofib}(\mathrm{F}^\infty \to X)$, then the completion $\widehat{\mathrm{F}^\star}$ is a complete filtration on $\hat{X}$.

By taking graded pieces, we will lose some information contained in filtrations, for example, the differentials:

There is an analogy of cochain complexes in the graded structure of filtrations. This was already known by Beilinson as in [Bei87] and in [[Lur17], §1.2.2] it was considered under the name of $\mathcal{J}$-complex. For this, we need $\mathcal{C}$ to be a stable ∞-category.

Let $\mathrm{F}^\star$ be a filtration over $\mathcal{C}$, the cofiber sequence

$$\mathrm{gr}^{s+1} \to \mathrm{gr}^{[s,s+2)} \to \mathrm{gr}^s \tag{2.1.9}$$

gives rise to a map $d^s : \mathrm{gr}^s \to \mathrm{gr}^{s+1}[1]$.

Proposition 2.1.9 $d^s \circ d^{s-1} \simeq 0$.

Proof. Immediately from the following diagram:

$$\begin{tikzcd}
\mathrm{gr}^{s+2} \arrow[r] \arrow[d, equal] & \mathrm{gr}^{[s+1,s+3)} \arrow[r] \arrow[d] & \mathrm{gr}^{s+1} \arrow[r] \arrow[d] & \mathrm{gr}^{s+2}[1] \arrow[d, equal] \\
\mathrm{gr}^{s+2} \arrow[r] \arrow[d] & \mathrm{gr}^{[s,s+3)} \arrow[r] \arrow[d] & \mathrm{gr}^{[s,s+2)} \arrow[r] \arrow[d] & \mathrm{gr}^{s+2}[1] \arrow[d] \\
\mathrm{gr}^{[s+2,s+2)} \simeq 0 \arrow[r] \arrow[d] & \mathrm{gr}^s \arrow[r, equal] \arrow[d] & \mathrm{gr}^s \arrow[r] \arrow[d, "d"'] & \mathrm{gr}^{[s+2,s+2)}[1] \simeq 0 \arrow[d] \\
\mathrm{gr}^{s+2}[1] \arrow[r] & \mathrm{gr}^{[s+1,s+3)}[1] \arrow[r] & \mathrm{gr}^{s+1}[1] \arrow[r, "d"] & \mathrm{gr}^{s+2}[2]
\end{tikzcd}$$

where all rows and columns are cofiber sequences. □

Inspired by this proposition, we may define a chain complex structure on pointed ∞-category $\mathcal{C}$. Let Ξ be the pointed 1-category with objects in $\mathbb{Z}_*$, the pointed integers, where $*$ is both initial and terminal. The morphisms of Ξ can be described as follows:

$$\mathrm{Hom}_\Xi(m,n) = \begin{cases} * & \text{if } n \neq m,\ m-1 \\ \mathrm{id}, * & \text{if } n = m \\ \delta, * & \text{if } n = m-1 \end{cases} \tag{2.1.10}$$

such that $\delta \circ \delta = *$.

Definition 2.1.10 Let $\mathcal{C}$ be a pointed ∞-category, the ∞-category of coherent cochain complexes $\mathrm{Ch}^\bullet(\mathcal{C})$ is defined to be the pointed functor category $\mathrm{Fun}_*(\Xi^{\mathrm{op}}, \mathcal{C})$, while the coherent chain complexes $\mathrm{Ch}_\bullet(\mathcal{C})$ are the covariant pointed functors $\mathrm{Fun}_*(\Xi, \mathcal{C})$.

Remark 2.1.11 If $\mathcal{A}$ is an abelian category in the usual sense, then $\mathrm{Ch}^\bullet(\mathcal{A})$ is just the category of cochain complexes on $\mathcal{A}$.

2.2 Beilinson t-structure

The following theorem is due to Ariotta. It relates a complete filtration with a coherent cochain complex in a stable ∞-category.

Theorem 2.2.1 *[[Ari21], Theorem 4.7] Let $\mathcal{C}$ be a stable ∞-category with sequential limits. There is a canonical categorical equivalence of complete filtrations and coherent cochain complexes:*

$$\mathrm{Fil}_c(\mathcal{C}) \simeq \mathrm{Ch}^\bullet(\mathcal{C}) \tag{2.2.1}$$

which sends a complete filtration $\mathrm{F}^\star$ to a cochain complex C with $C^n \simeq \mathrm{gr}^n\mathrm{F}^\star[n]$.

Definition 2.2.2 Let $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ be a t-structure on a stable ∞-category $\mathcal{C}$ with sequential limits. Consider the pointwise t-structure on $\mathrm{Ch}^\bullet(\mathcal{C})$, by (2.2.1) we can thus define a t-structure $(\mathrm{Fil}_c(\mathcal{C})^{\mathrm{B}}_{\geq 0}, \mathrm{Fil}_c(\mathcal{C})^{\mathrm{B}}_{\leq 0})$ on $\mathrm{Fil}_c(\mathcal{C})$, which is called the Beilinson t-structure on $\mathrm{Fil}_c(\mathcal{C})$.

Remark 2.2.3 Unfold the equivalence in Theorem 2.2.1, the connective objects in Beilinson t -structure are just those complete filtrations $\mathrm{F}^\star$ such that $\mathrm{gr}^n\mathrm{F}^\star \in \mathcal{C}_{\geq -n}$, and coconnective objects are those $\mathrm{F}^\star$ such that $\mathrm{gr}^n\mathrm{F}^\star \in \mathcal{C}_{\leq -n}$. The heart of the Beilinson t-structure is $\mathrm{Ch}^\bullet(\mathcal{C})^\heartsuit \simeq \mathrm{Ch}^\bullet(\mathcal{C}^\heartsuit)$.

Historically, in [Bei87] Beilinson tried to define the t-structure on non-complete filtration $\mathrm{Fil}(\mathcal{C})$ by declaring $\mathrm{Fil}(\mathcal{C})^{\mathrm{B}}_{\geq 0}$ as the full subcategory of those filtrations $\mathrm{F}^\star$ with $\mathrm{gr}^n\mathrm{F}^\star \in \mathcal{C}_{\geq -n}$. This is however better to handle the non-complete case like the following:

Construction 2.2.4 For incomplete filtrations $\mathrm{F}^\star \in \mathrm{Fil}(\mathcal{C})$, we may consider the adjunction pairs

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "i"] & & \mathrm{Fil}(\mathcal{C}) \arrow[ll, bend right, "i_{\mathrm{L}}"'] \arrow[ll, bend left, "i_{\mathrm{R}}"] \arrow[rr, bend left, "j_{\mathrm{L}}"] \arrow[rr, phantom, "\perp"] & & \mathrm{Fil}_c(\mathcal{C}) \arrow[ll, bend left, "j"]
\end{tikzcd}$$

where i the constant filtration functor and j the inclusion. From the previous section we know

$$\begin{aligned} i_{\mathrm{L}}(\mathrm{F}^\star) &= |\mathrm{F}^\star| \\ i_{\mathrm{R}}(\mathrm{F}^\star) &= \mathrm{F}^\infty \\ j_L(\mathrm{F}^\star) &= \widehat{\mathrm{F}^\star}. \end{aligned} \tag{2.2.2}$$

Now let $(\mathcal{C}^t_{\geq 0}, \mathcal{C}^t_{\leq 0}) := (\mathcal{C}, 0)$ be the trivial t-structure on $\mathcal{C}$, we can define the glued t-structure on $\mathrm{Fil}(\mathcal{C})$ given by

$$\begin{aligned} \mathrm{Fil}(\mathcal{C})_{\geq 0} &= \{\mathrm{F}^\star \in \mathrm{Fil}(\mathcal{C}) : j_{\mathrm{L}}(\mathrm{F}^\star) \in \mathrm{Fil}_c(\mathcal{C})^{\mathrm{B}}_{\geq 0}, i_{\mathrm{L}}(\mathrm{F}^\star) \in \mathcal{C}^t_{\geq 0}\} \\ \mathrm{Fil}(\mathcal{C})_{\leq 0} &= \{\mathrm{F}^\star \in \mathrm{Fil}(\mathcal{C}) : j_{\mathrm{L}}(\mathrm{F}^\star) \in \mathrm{Fil}_c(\mathcal{C})^{\mathrm{B}}_{\leq 0}, i_{\mathrm{R}}(\mathrm{F}^\star) \in \mathcal{C}^t_{\leq 0}\}. \end{aligned} \tag{2.2.3}$$

Explain in words, a connective object in $\mathrm{Fil}(\mathcal{C})$ is just a filtration whose completion is connective in $\mathrm{Fil}_c(\mathcal{C})$ with respect to Beilinson t-structure and a coconnective object is a complete filtration which is coconnective in $\mathrm{Fil}_c(\mathcal{C})$, i.e. $\mathrm{Fil}(\mathcal{C})_{\leq 0} \simeq \mathrm{Fil}_c(\mathcal{C})^{\mathrm{B}}_{\leq 0}$.

Remark 2.2.5 In our setting we will call this construction on $\mathrm{Fil}(\mathcal{C})$ also the Beilinson t-structure. But be aware of the difference between this t-structure and the one defined in [Bei87]. Only under the assumption of right separateness of $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ we can prove the equivalence of these two structures, for a proof see [[Ari21], Theorem 5.11].

We consider the following composition of functors

$$\mathrm{Ch}^\bullet(\mathcal{C}^\heartsuit) \hookrightarrow \mathrm{Fil}_c(\mathcal{C})^{\mathrm{B}}_{\geq 0} \hookrightarrow \mathrm{Fil}_c(\mathcal{C}) \hookrightarrow \mathrm{Fil}(\mathcal{C}) \xrightarrow{|-|} \mathcal{C} \tag{2.2.4}$$

which sends a coherent cochain complex $X^\bullet$ to $|\sigma^\star X^\bullet|$, where $\sigma^\star X^\bullet$ is the truncation filtration defined in Example 2.1.6. For simplicity we just write $|X^\bullet|$.

Lemma 2.2.6 *Let $\mathcal{C}$ be a stable ∞-category with sequential limits and colimits, let $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ be a t-structure on $\mathcal{C}$. For $X^\bullet \in \mathrm{Ch}^\bullet(\mathcal{C})$ a coherent cochain complex, we have*

$$H^{-n}(X^\bullet) \cong \pi_n(|X^\bullet|) \tag{2.2.5}$$

Proof. By shifting we reduce to the case $n = 0$. Recall the graded piece on $\sigma^\star X$ is given by $\mathrm{gr}^n \sigma^\star X = X^n[-n]$ for $X \in \mathrm{Ch}^\bullet(\mathcal{C})$. We can construct the following short exact sequence in $\mathcal{C}^\heartsuit$:

$$0 \to \pi_0 \mathrm{gr}^{[0,2)} \to X^0 \xrightarrow{d} X^1 \to \pi_{-1}\mathrm{gr}^{[0,2)} \to 0. \tag{2.2.6}$$

This implies that we can identify $\pi_0 \mathrm{gr}^{[0,2)}$ with the cocycle Z^0 of degree 0. On the other hand, the fiber sequence $\mathrm{gr}^{[0,2)} \to \mathrm{gr}^{[-1,2)} \to \mathrm{gr}^{-1}$ yields an exact sequence

$$X^{-1} \xrightarrow{d} Z^0 \to \pi_0 \mathrm{gr}^{[-1,2)} \to 0. \tag{2.2.7}$$

This shows $\pi_0 \mathrm{gr}^{[-1,2)} \cong H^0(X^\bullet)$.
Next we notice $\tau_{\geq 0}\mathrm{gr}^{[-1,s+1)} \to \tau_{\geq 0}\mathrm{gr}^{[-1,s)}$ is an equivalence for $s \geq 2$ by applying $\tau_{\geq 0}$ to the fiber sequence

$$\mathrm{gr}^{[-1,s+1)} \to \mathrm{gr}^{[-1,s)} \to X^s[-s+1]. \tag{2.2.8}$$

Passing to $\lim_s$ yields

$$\tau_{\geq 0}\mathrm{gr}^{[-1,\infty)} \simeq \lim_s \tau_{\geq 0}\mathrm{gr}^{[-1,s)} \simeq \tau_{\geq 0}\mathrm{gr}^{[-1,2)} \tag{2.2.9}$$

and $\pi_0 \mathrm{gr}^{[-1,\infty)} \cong H^0(X^\bullet)$. Moreover for $t \geq 1$, $\tau_{\leq 0}\mathrm{gr}^{[-t,\infty)} \to \tau_{\leq 0}\mathrm{gr}^{[-t-1,\infty)}$ is an equivalence using the cofiber sequence

$$\mathrm{gr}^{-t-1}[-1] \to \mathrm{gr}^{[-t,\infty)} \to \mathrm{gr}^{[-t-1,\infty)}. \tag{2.2.10}$$

Passing to colim_t yields

$$\tau_{\leq 0}X \simeq \mathrm{colim}_t \tau_{\leq 0}\mathrm{gr}^{[-t,\infty)} \simeq \tau_{\leq 0}\mathrm{gr}^{[-1,\infty)}. \tag{2.2.11}$$

As $\pi_0 \simeq \tau_{\leq 0} \circ \tau_{\geq 0}$, this completes the proof. □

The following lemma will be helpful in the next section.

Lemma 2.2.7 *If* $\mathrm{F}^\star X$ *is a complete filtration on* X*, then each* $\tau^{\mathrm{B}}_{\geq n}(\mathrm{F}^\star X)$ *is complete.*

Proof. As there is a cofiber sequence in $\mathrm{Fil}(\mathcal{C})$

$$\tau^{\mathrm{B}}_{\geq n}(\mathrm{F}^\star X) \to \mathrm{F}^\star X \to \tau^{\mathrm{B}}_{\leq n-1}(\mathrm{F}^\star X) \tag{2.2.12}$$

the lemma now follows from the assumption and the fact $\mathrm{Fil}(\mathcal{C})^{\mathrm{B}}_{\leq 0} \simeq \mathrm{Fil}_c(\mathcal{C})^{\mathrm{B}}_{\leq 0}$. □

2.3 Décalage

In this section we fix $\mathcal{C}$ a stable ∞-category with sequential limits and colimits, equipped with a t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$.

Inspired by Deligne's décalage operation [Del71], Levine considered a naïve version of décalage of cosimplicial spectra in [[Lev15], §6], in order to give a comparison between the motivic Atiyah-Hirzebruch sequence and the Adams-Novikov sequence for the complex cobordism spectrum. Later in [BMS19] this was again defined using Beilinson t-structure (cf. [[BMS19], Proposition 5.8]). The equivalence of this definition with Deligne's construction was shown in [Ant24]. We will mainly follow the last reference in the section.

As discussed in the previous section, a filtration $\mathrm{F}^\star$ gives rise to a coherent cochain complex

$$\dots \to \mathrm{gr}^{-s-1}\mathrm{F}^\star[-s-1] \to \mathrm{gr}^{-s}\mathrm{F}^\star[-s] \to \mathrm{gr}^{-s+1}\mathrm{F}^\star[-s+1] \to \dots \tag{2.3.1}$$

Apply the π_t functor and after suitable suspension yields a coherent cochain complex in the heart $\mathcal{C}^\heartsuit$ of $\mathcal{C}$

$$\dots \to \pi_{s+t+1}\mathrm{gr}^{-s-1}\mathrm{F}^\star \to \pi_{s+t}\mathrm{gr}^{-s}\mathrm{F}^\star \to \pi_{s+t-1}\mathrm{gr}^{-s+1}\mathrm{F}^\star \to \dots \tag{2.3.2}$$

Definition 2.3.1 Let $\mathrm{F}^\star$ be a filtration. Consider the Whitehead tower with respect to the Beilinson t-structure on $\mathrm{Fil}(\mathcal{C})$

$$\dots \to \tau^{\mathrm{B}}_{\geq n+1}\mathrm{F}^\star \to \tau^{\mathrm{B}}_{\geq n}\mathrm{F}^\star \to \tau^{\mathrm{B}}_{\geq n-1}\mathrm{F}^\star \to \dots \tag{2.3.3}$$

By taking the realization, we get a new filtered object of $\mathcal{C}$

$$\dots \to |\tau^{\mathrm{B}}_{\geq n+1}\mathrm{F}^\star| \to |\tau^{\mathrm{B}}_{\geq n}\mathrm{F}^\star| \to |\tau^{\mathrm{B}}_{\geq n-1}\mathrm{F}^\star| \to \dots \tag{2.3.4}$$

This is called the décalage of $\mathrm{F}^\star$, and we denote $\mathrm{Dec}^\bullet(\mathrm{F}^\star)$. If $\mathrm{F}^\star$ is a filtration on X, Since we have natural maps $\tau^{\mathrm{B}}_{\geq n}\mathrm{F}^\star \to \mathrm{F}^\star$, we then have a map

$$|\tau^{\mathrm{B}}_{\geq n}\mathrm{F}^\star| \to |\mathrm{F}^\star| \to X \tag{2.3.5}$$

hence $\mathrm{Dec}^\bullet(\mathrm{F}^\star)$ is a filtration on X.

It's immediate to see the graded pieces of $\mathrm{Dec}^\bullet(\mathrm{F}^\star)$ are given by

$$\mathrm{gr}^n\mathrm{Dec}^\bullet(\mathrm{F}^\star) \simeq |\pi^{\mathrm{B}}_n(\mathrm{F}^\star)|[n] \tag{2.3.6}$$

as $\tau^{\mathrm{B}}_{\geq n+1}(\mathrm{F}^\star) \to \tau^{\mathrm{B}}_{\geq n}(\mathrm{F}^\star) \to \pi^{\mathrm{B}}_n(\mathrm{F}^\star)$ is a cofiber sequence.

Remark 2.3.2 The cochain complex of homotopy groups of $F^\star$ with respect to the Beilinson t-structure is denoted $\pi_n^{\text{B}}(F^\star)^\bullet$, which by (2.3.2) is

$$\dots \to \pi_{n+1}\mathrm{gr}^{-1} \to \pi_n \mathrm{gr}^0 \to \pi_{n-1}\mathrm{gr}^1 \to \dots \tag{2.3.7}$$

this is precisely the n-th vertical column of the E_1-page of the spectral sequence associated to $F^\star$ as in [Lur17]. This illustrates another interpretation of the décalage functor: the n-fold suspension of realization of $\pi_n^{\text{B}}(F^\star)^\bullet$

Before proceeding to higher pages of spectral sequences, we give some important examples of the décalage functor.

Example 2.3.3 If $F^\star X$ is the constant filtration on X, then each $\tau_{\geq 0}^{\text{B}}(F^\star X)$ is a constant filtration and so is $\mathrm{Dec}^\bullet(F^\star X)$ on X.

We already know that each cofiber sequence $F^\star X' \to F^\star X \to F^\star X''$ induces a long exact sequence in $\mathrm{Ch}^\bullet(\mathcal{C}^\heartsuit)$:

$$\dots \to \pi_{n+1}^{\text{B}}(F^\star X'') \to \pi_n^{\text{B}}(F^\star X') \to \pi_n^{\text{B}}(F^\star X) \to \pi_n^{\text{B}}(F^\star X'') \to \pi_{n-1}^{\text{B}}(F^\star X') \to \dots \tag{2.3.8}$$

This in general does not break into short exact sequences as the connecting homomorphism is not trivial. However, the following example is an exception.

Example 2.3.4 Let $F^\star$ be a filtration. We have a cofiber sequence of filtrations

$$\mathrm{gr}^{[b,c)} \to \mathrm{gr}^{[a,c)} \to \mathrm{gr}^{[a,b)} \tag{2.3.9}$$

for $-\infty \leq a \leq b \leq c \leq \infty$. Then this induces a short exact sequence

$$0 \to \pi_0^{\text{B}}\big(\mathrm{gr}^{[b,c)}\big) \to \pi_0^{\text{B}}\big(\mathrm{gr}^{[a,c)}\big) \to \pi_0^{\text{B}}\big(\mathrm{gr}^{[a,b)}\big) \to 0 \tag{2.3.10}$$

Indeed, the middle term expands to

$$\dots \to 0 \to \pi_{-c+1}F^{c-1} \to \dots \to \pi_{-a-1}F^{a+1} \to \pi_{-a}F^a \to 0 \to \dots \tag{2.3.11}$$

while the left and right terms are just truncations of this cochain complex.

Example 2.3.5 By the exactness of geometric realizations and previous example

$$\mathrm{Dec}^\bullet\big(\mathrm{gr}^{[b,c)}\big) \to \mathrm{Dec}^\bullet\big(\mathrm{gr}^{[a,c)}\big) \to \mathrm{Dec}^\bullet\big(\mathrm{gr}^{[a,b)}\big) \tag{2.3.12}$$

is a cofiber sequence of filtered objects.

Let $\mathrm{ins}^s : \mathcal{C} \to \mathrm{Fil}(\mathcal{C})$ be the left Kan extension of the constant functor $\mathcal{C} \to \mathrm{Fil}(\mathcal{C})$ along the inclusion $\{s\} \hookrightarrow \mathbb{Z}^{\mathrm{op}}$. Concretely, we have

$$F^i \mathrm{ins}^s X \simeq \begin{cases} 0 & \text{if } i > s \\ X & \text{if } i \leq s \end{cases} \tag{2.3.13}$$

and all transition maps are the identity.

Lemma 2.3.6 *The functor* $\mathrm{ins}^s : \mathcal{C} \to \mathrm{Fil}(\mathcal{C})$ *is t-exact with respect to the Beilinson t-structure.*

Proof. As ins^s lands in $\mathrm{Fil}_c(\mathcal{C})$, it's enough to show it is t-exact on the coherent cochain complex by Theorem 2.2.1. Under this equivalence, $\mathrm{ins}^s X$ is the complex given by $X[s]$ in degree s and 0 elsewhere and is surely t-exact. □

Corollary 2.3.7 *Let* $\mathrm{F}^\star$ *be a filtration on* X*. The filtration on* $\mathrm{Dec}^\bullet(\mathrm{gr}^s\mathrm{F}^\star X)$ *is equivalent to* $\tau_{\geq -s+\bullet}\mathrm{gr}^s\mathrm{F}^\star X$*. In particular, one has*

$$\begin{aligned} \mathrm{gr}^{[r,\infty)}(\mathrm{Dec}^\bullet(\mathrm{gr}^s\mathrm{F}^\star X)) &\simeq \tau_{\geq -s+r}\mathrm{gr}^s\mathrm{F}^\star X \\ \mathrm{gr}^{(-\infty,r]}(\mathrm{Dec}^\bullet(\mathrm{gr}^s\mathrm{F}^\star X)) &\simeq \tau_{\leq -s+r}\mathrm{gr}^s\mathrm{F}^\star X \end{aligned} \tag{2.3.14}$$

Proof. It suffices to consider the case $s = 0$, in which we have

$$\tau^{\mathrm{B}}_{\geq\bullet}(\mathrm{gr}^0\mathrm{F}^\star X) = \tau^{\mathrm{B}}_{\geq\bullet}(\mathrm{ins}^0 X) \simeq \mathrm{ins}^0(\tau_{\geq\bullet}X) \tag{2.3.15}$$

by the previous lemma. Taking colimits finishes the proof. □

If we are not focusing on the filtration property of décalage, we will just write $\mathrm{Dec}(\mathrm{F}^\star)$. The most important usage of décalage functor is to build spectral sequences.

Definition 2.3.8 The E_r-page of the spectral sequence associated to a filtration $\mathrm{F}^\star$ is defined inductively to be

$$\begin{aligned} E_1^{s,t}(\mathrm{F}^\star) &:= \pi_{s+t}\mathrm{gr}^{-s}\mathrm{F}^\star \\ E_{r+1}^{s,t}(\mathrm{F}^\star) &:= E_r^{-t,s+2t}(\mathrm{Dec}(\mathrm{F}^\star)) \end{aligned} \tag{2.3.16}$$

together with the differential from (2.3.2):

$$\begin{aligned} d_1^{s,t} : \pi_{s+t}\mathrm{gr}^{-s}\mathrm{F}^* &\to \pi_{s+t-1}\mathrm{gr}^{-s+1}\mathrm{F}^* \\ d_{r+1}^{s,t} &:= d_r^{-t,s+2t}. \end{aligned} \tag{2.3.17}$$

Lemma 2.3.9 *The construction in Definition 2.3.8 indeed gives a spectral sequence.*

Proof. We need to show the E_{r+1}-page is the cohomology of E_r-page. We do it by induction on r and it's enough to check it by $r = 1$.

$$\begin{aligned} E_2^{s,t}(\mathrm{F}^\star) = E_1^{-t,s+2t}(\mathrm{Dec}(\mathrm{F}^\star)) &\cong \pi_{s+t}\mathrm{gr}^t(\mathrm{Dec}(\mathrm{F}^\star)) \cong \pi_{s+t}(|\pi_t^{\mathrm{B}}(\mathrm{F}^\star)|[t]) \\ \cong \pi_s(|\pi_t^{\mathrm{B}}(\mathrm{F}^\star)|) &\cong H^{-s}(\pi_t^{\mathrm{B}}(\mathrm{F}^\star)^\bullet) =: H^s\left(E_1^{\bullet,t}(\mathrm{F}^\star)\right) \end{aligned} \tag{2.3.18}$$

where the last isomorphism comes from Lemma 2.2.6. □

We introduce another construction of spectral sequences due to [[Lur17], §1.2.2]. The advantage of this construction is that it does not assume the ∞-category $\mathcal{C}$ admits sequential limits and colimits.

Construction 2.3.10 Let $r \geq 1$, consider the commutative square

$$\begin{tikzcd} \mathrm{F}^{-s+r} \arrow[r] \arrow[d] & \mathrm{F}^{-s} \arrow[d] \\ \mathrm{F}^{-s+1} \arrow[r] & \mathrm{F}^{-s-r+1} \end{tikzcd}$$

By taking the cofibers we get a natural map $\mathrm{gr}^{[-s,-s+r)} \to \mathrm{gr}^{[-s-r+1,-s+1)}$, this fits into the following commutative diagram

$$\begin{tikzcd} \mathrm{gr}^{[-s,-s+2r)} \arrow[r] \arrow[d] & \mathrm{gr}^{[-s+r,-s+2r)} \arrow[r] \arrow[d] & \mathrm{gr}^{[-s,-s+r)} \arrow[d] \\ \mathrm{gr}^{[-s-r+1,-s+r+1)} \arrow[r] & \mathrm{gr}^{[-s+1,-s+r+1)} \arrow[r] & \mathrm{gr}^{[-s-r+1,-s+1)} \end{tikzcd}$$

where the horizontal lines are cofiber sequences.

We set $E_r^{s,t}(\mathrm{F}^\star) := \mathrm{im}\big(\pi_{s+t}\mathrm{gr}^{[-s,-s+r)} \to \pi_{s+t}\mathrm{gr}^{[-s-r+1,-s+1)}\big)$. To construct the differential, we firstly notice that the heart of a t-structure is (the nerve of) an abelian category, thus by epi-mono factorization we have

$$\begin{tikzcd} \pi_{s+t}\mathrm{gr}^{[-s,-s+r)} \arrow[r, two heads] \arrow[d] & E_r^{s,t} \arrow[r, hook] \arrow[d, dashed, "d_r^{s,t}"'] & \pi_{s+t}\mathrm{gr}^{[-s-r+1,-s+1)} \arrow[d] \\ \pi_{s+t-1}\mathrm{gr}^{[-s+r,-s+2r)} \arrow[r, two heads] & E_r^{s-r,t+r-1} \arrow[r, hook] & \pi_{s+t-1}\mathrm{gr}^{[-s+1,-s+r+1)} \end{tikzcd}$$

where the outer square has homotopy boundary maps as vertical arrows and naturally commutes.

On the E_1-page, we have

$$\mathrm{im}\left(\pi_{s+t}\mathrm{gr}^{-s} \xrightarrow{\mathrm{id}} \pi_{s+t}\mathrm{gr}^{-s}\right) = E_1^{s,t}(\mathrm{F}^\star) \tag{2.3.19}$$

and the differential coincides with the one in (2.3.2).

Lemma 2.3.11 *[[Lur17], Proposition. 1.2.2.7] The E_r-page together with differentials in Construction 2.3.10 is a spectral sequence.*

The main theorem of this section is the following comparison theorem. In particular, this proof will be performed inductively and it already implies Lemma 2.3.11.

Theorem 2.3.12 *[[Ant24], Lemma 4.24] For $r \geq 1$, there's a natural isomorphism*

$$E_1^{-(r-1)s-rt,rs+(r+1)t}\big(\mathrm{Dec}^{(r)}(\mathrm{F}^\star)\big) \cong E_{r+1}^{s,t}(\mathrm{F}^\star) \tag{2.3.20}$$

compatible with differentials, where $E_{r+1}^{s,t}(\mathrm{F}^\star)$ denotes Lurie's construction.

Proof. Inductively we need to show

$$E_r^{-t,s+2t}(\mathrm{Dec}(\mathrm{F}^\star)) \cong E_{r+1}^{s,t}(\mathrm{F}^\star) \tag{2.3.21}$$

We consider the following diagram arising from the functoriality of décalage, for simplicity we denote $\mathrm{gr}_{\mathrm{D}}^{[a,b)}$ for $\mathrm{gr}^{[a,b)}(\mathrm{Dec}(-))$ and $\mathrm{gr}_{\mathrm{F}}^{[a,b)}$ for $\mathrm{gr}^{[a,b)}\mathrm{F}^{\star}$ and we set $s = 0 = t$.

$$\begin{tikzcd}
\pi_0\mathrm{gr}_{\mathrm{F}}^{[0,r+1)} \arrow[d, two heads] & \pi_0\mathrm{gr}_{\mathrm{D}}^{[0,\infty)}\mathrm{gr}_{\mathrm{F}}^{[0,r+1)} \arrow[l, "1", "\cong"'] \arrow[r, two heads, "2"] \arrow[d, two heads] & \pi_0\mathrm{gr}_{\mathrm{D}}^{[0,r)}\mathrm{gr}_{\mathrm{F}}^{[0,r+1)} \arrow[d, two heads] & \pi_0\mathrm{gr}_{\mathrm{D}}^{[0,r)}\mathrm{gr}_{\mathrm{F}}^{[0,\infty)} \arrow[l, "3", "\cong"'] \arrow[r, two heads, "4"] \arrow[d, two heads] & \pi_0\mathrm{gr}_{\mathrm{D}}^{[0,r)}\mathrm{F}^{\star} \arrow[d, two heads] \\
E_{r+1}^{0,0}(\mathrm{F}^{\star}) & A & B & C & E_{r}^{0,0}(\mathrm{Dec}(\mathrm{F}^{\star})) \\
\pi_0\mathrm{gr}_{\mathrm{F}}^{[-r,1)} \arrow[u, hook] & \pi_0\mathrm{gr}_{\mathrm{D}}^{[-r+1,\infty)}\mathrm{gr}_{\mathrm{F}}^{[-r,1)} \arrow[l, hook', "5"] \arrow[r, "6", "\cong"'] \arrow[u, hook] & \pi_0\mathrm{gr}_{\mathrm{D}}^{[-r+1,1)}\mathrm{gr}_{\mathrm{F}}^{[-r,1)} \arrow[u, hook] & \pi_0\mathrm{gr}_{\mathrm{D}}^{[-r+1,1)}\mathrm{gr}_{\mathrm{F}}^{[-r,\infty)} \arrow[l, hook', "7"] \arrow[r, "8", "\cong"'] \arrow[u, hook] & \pi_0\mathrm{gr}_{\mathrm{D}}^{[-r+1,1)}\mathrm{F}^{\star} \arrow[u, hook]
\end{tikzcd}$$

We will justify the marked morphisms separately.

1. The colimit of the fiber sequence arising from Beilinson t-structure
$$\tau_{\geq 0}^{\mathrm{B}}\mathrm{gr}_{\mathrm{F}}^{[0,r+1)} \to \mathrm{gr}_{\mathrm{F}}^{[0,r+1)} \to \tau_{\leq -1}^{\mathrm{B}}\mathrm{gr}_{\mathrm{F}}^{[0,r+1)} \tag{2.3.22}$$
is by definition
$$\mathrm{gr}_{\mathrm{D}}^{[0,\infty)}\mathrm{gr}_{\mathrm{F}}^{[0,r+1)} \to \mathrm{gr}_{\mathrm{F}}^{[0,r+1)} \to \mathrm{gr}_{\mathrm{D}}^{(-\infty,-1]}\mathrm{gr}_{\mathrm{F}}^{[0,r+1)} \tag{2.3.23}$$
Use Example 2.3.5 we see that $\mathrm{gr}_{\mathrm{D}}^{(-\infty,-1]}\mathrm{gr}_{\mathrm{F}}^{[0,r+1)}$ has a finite filtration with graded piece $\mathrm{gr}_{\mathrm{D}}^{(-\infty,-1]}\mathrm{gr}_{\mathrm{F}}^{s} \simeq \tau_{\leq -s-1}\mathrm{gr}_{\mathrm{F}}^{s}, 0 \leq s \leq r$ by Corollary 2.3.7. This tells us $\mathrm{gr}_{\mathrm{D}}^{(-\infty,-1]}\mathrm{gr}_{\mathrm{F}}^{[0,r+1)}$ is in $\mathcal{C}_{\leq -1}$ and thus 1 is an isomorphism.
2. Like the setting above, $\mathrm{gr}_{\mathrm{D}}^{[0,\infty)}\mathrm{gr}_{\mathrm{F}}^{[0,r+1)}$ has a finite filtration with graded piece $\mathrm{gr}_{\mathrm{D}}^{[0,\infty)}\mathrm{gr}_{\mathrm{F}}^{s} \simeq \tau_{\geq -s}\mathrm{gr}_{\mathrm{F}}^{s}$ and similarly $\mathrm{gr}_{\mathrm{D}}^{[0,r)}\mathrm{gr}_{\mathrm{F}}^{[0,r+1)}$ is finitely filtered. The fiber sequence
$$\mathrm{gr}_{\mathrm{D}}^{[r,\infty)}\mathrm{gr}_{\mathrm{F}}^{[0,r+1)} \to \mathrm{gr}_{\mathrm{D}}^{[0,\infty)}\mathrm{gr}_{\mathrm{F}}^{[0,r+1)} \to \mathrm{gr}_{\mathrm{D}}^{[0,r)}\mathrm{gr}_{\mathrm{F}}^{[0,r+1)} \tag{2.3.24}$$
has a finite filtered fiber with each graded piece $\mathrm{gr}_{\mathrm{D}}^{[r,\infty)}\mathrm{gr}_{\mathrm{F}}^{s} \simeq \tau_{\geq -s+r}\mathrm{gr}_{\mathrm{F}}^{s}, 0 \leq s \leq r$. Therefore the fiber is connective and the map 2 under π_0 is an epimorphism.
3. The fiber sequence
$$\mathrm{gr}_{\mathrm{D}}^{[0,r)}\mathrm{gr}_{\mathrm{F}}^{[r+1,\infty)} \to \mathrm{gr}_{\mathrm{D}}^{[0,r)}\mathrm{gr}_{\mathrm{F}}^{[0,\infty)} \to \mathrm{gr}_{\mathrm{D}}^{[0,r)}\mathrm{gr}_{\mathrm{F}}^{[0,r+1)} \tag{2.3.25}$$
has a finite filtered fiber with associated graded piece $|\pi_s^{\mathrm{B}}\mathrm{gr}_{\mathrm{F}}^{[r+1,\infty)}[s]|$ for $0 \leq s < r$. The cochain complex $\pi_s^{\mathrm{B}}\mathrm{gr}_{\mathrm{F}}^{[r+1,\infty)}$ is of the form
$$\ldots \to 0 \to \pi_{-r-1+s}\mathrm{gr}_{\mathrm{F}}^{r+1} \to \pi_{-r-1+s-1}\mathrm{gr}_{\mathrm{F}}^{r+2} \to \ldots \tag{2.3.26}$$
with $\pi_{-r-1+s}\mathrm{gr}_{\mathrm{F}}^{r+1}$ put on the cohomological degree $r+1$. Now as $s < r+1$, $H^s = 0$ and by Lemma 2.2.6,
$$\pi_{-a}|\pi_s^{\mathrm{B}}\mathrm{gr}_{\mathrm{F}}^{[r+1,\infty)}[s]| \cong H^{a+s}\left(\pi_s^{\mathrm{B}}\mathrm{gr}_{\mathrm{F}}^{[r+1,\infty)}\right) = 0 \tag{2.3.27}$$
for $a < r+1-s$. In particular, $\pi_1 = 0$, hence the fiber is 1-connective and 3 is an isomorphism.
4. The cofiber sequence
$$\mathrm{gr}_{\mathrm{D}}^{[0,r)}\mathrm{gr}_{\mathrm{F}}^{[0,\infty)} \to \mathrm{gr}_{\mathrm{D}}^{[0,r)}\mathrm{F}^{\star} \to \mathrm{gr}_{\mathrm{D}}^{[0,r)}\mathrm{gr}_{\mathrm{F}}^{(-\infty,-1]} \tag{2.3.28}$$
has a connective cofiber by the same nature of 3. Again by Lemma 2.2.6,

$$\pi_{-a}|\pi_s^{\text{Б}}\mathrm{gr}_{\mathrm{F}}^{(-\infty,-1]}[s]| \cong H^{a+s}\left(\pi_s^{\text{Б}}\mathrm{gr}_{\mathrm{F}}^{(-\infty,-1]}\right) = 0 \tag{2.3.29}$$

for $a > -1-s$. In particular $\pi_0 = 0$ and 4 is an epimorphism.

5. Same as in case of 1, but as $\mathrm{gr}_{\mathrm{D}}^{(-\infty,-r]}\mathrm{gr}_{\mathrm{F}}^{s} \simeq \tau_{\leq -s-r}\mathrm{gr}_{\mathrm{F}}^{s}, -r \leq s \leq 0$, we conclude that $\mathrm{gr}_{\mathrm{D}}^{(-\infty,-r]}\mathrm{gr}_{\mathrm{F}}^{[-r,1)}$ is in $\mathcal{C}_{\leq 0}$ and 5 is a monomorphism.
6. Same as in case of 2, the fiber is however 1-connective and 6 must be an isomorphism.
7. Repeat the arguments in 3, we have the fiber sequence
$$\mathrm{gr}_{\mathrm{D}}^{[-r+1,1)}\mathrm{gr}_{F}^{[1,\infty)} \to \mathrm{gr}_{\mathrm{D}}^{[-r+1,1)}\mathrm{gr}_{\mathrm{F}}^{[-r,\infty)} \to \mathrm{gr}_{\mathrm{D}}^{[-r+1,1)}\mathrm{gr}_{\mathrm{F}}^{[-r,1)} \tag{2.3.30}$$
and the fiber has trivial π_0 hence the map 7 is a monomorphism.
8. As in 4, the cofiber of
$$\mathrm{gr}_{\mathrm{D}}^{[-r+1,1)}\mathrm{gr}_{\mathrm{F}}^{[-r,\infty)} \to \mathrm{gr}_{\mathrm{D}}^{[-r+1,1)}\mathrm{F}^\star \to \mathrm{gr}_{\mathrm{D}}^{[-r+1,1)}\mathrm{gr}_{\mathrm{F}}^{(-\infty,-r-1]} \tag{2.3.31}$$
is 1-connective and 8 is an isomorphism.

Thus by standard arguments we have a chain of isomorphisms

$$E_{r+1}^{0,0}(\mathrm{F}^\star) \leftarrow A \to B \leftarrow C \to E_r^{0,0}(\mathrm{Dec}(\mathrm{F}^\star)) \tag{2.3.32}$$

It remains to show this isomorphism is compatible with differentials staring from $(0,0)$. For this, we firstly consider the following commutative diagram raising from fiber and cofiber sequences

$$\begin{tikzcd}
\mathrm{gr}_{\mathrm{F}}^{[r+1,2r+2)} \arrow[d, "\mathrm{id}"'] & \mathrm{gr}_{\mathrm{D}}^{[r,\infty)}\mathrm{gr}_{\mathrm{F}}^{[r+1,2r+2)} \arrow[l] \arrow[r] \arrow[d] & \mathrm{gr}_{\mathrm{D}}^{[r,2r)}\mathrm{gr}_{\mathrm{F}}^{[r+1,2r+2)} \arrow[d] & \mathrm{gr}_{\mathrm{D}}^{[r,2r)}\mathrm{gr}_{\mathrm{F}}^{[r+1,\infty)} \arrow[l] \arrow[r] \arrow[d] & \mathrm{gr}_{\mathrm{D}}^{[r,2r)}\mathrm{F}^\star \arrow[d, "\mathrm{id}"'] \\
\mathrm{gr}_{\mathrm{F}}^{[r+1,2r+2)} \arrow[d] & \mathrm{gr}_{\mathrm{D}}^{[0,\infty)}\mathrm{gr}_{\mathrm{F}}^{[r+1,2r+2)} \arrow[l] \arrow[r] \arrow[d] & A \arrow[d] & \mathrm{gr}_{\mathrm{D}}^{[r,2r)}\mathrm{gr}_{\mathrm{F}}^{[0,\infty)} \arrow[l] \arrow[r] \arrow[d] & \mathrm{gr}_{\mathrm{D}}^{[r,2r)}\mathrm{F}^\star \arrow[d] \\
\mathrm{gr}_{\mathrm{F}}^{[0,2r+2)} \arrow[d] & \mathrm{gr}_{\mathrm{D}}^{[0,\infty)}\mathrm{gr}_{\mathrm{F}}^{[0,2r+2)} \arrow[l] \arrow[r] \arrow[d] & \mathrm{gr}_{\mathrm{D}}^{[0,2r)}\mathrm{gr}_{\mathrm{F}}^{[0,2r+2)} \arrow[d] & \mathrm{gr}_{\mathrm{D}}^{[0,2r)}\mathrm{gr}_{\mathrm{F}}^{[0,\infty)} \arrow[l] \arrow[r] \arrow[d] & \mathrm{gr}_{\mathrm{D}}^{[0,2r)}\mathrm{F}^\star \arrow[d] \\
\mathrm{gr}_{\mathrm{F}}^{[0,r+1)} & \mathrm{gr}_{\mathrm{D}}^{[0,\infty)}\mathrm{gr}_{\mathrm{F}}^{[0,r+1)} \arrow[l] \arrow[r] & \mathrm{gr}_{\mathrm{D}}^{[0,r)}\mathrm{gr}_{\mathrm{F}}^{[0,r+1)} & \mathrm{gr}_{\mathrm{D}}^{[0,r)}\mathrm{gr}_{\mathrm{F}}^{[0,\infty)} \arrow[l] \arrow[r] & \mathrm{gr}_{\mathrm{D}}^{[0,r)}\mathrm{F}^\star
\end{tikzcd}$$

A exists since the lower 3×5-diagram has exact columns and is set to be the cofiber. The map $\mathrm{gr}_{\mathrm{D}}^{[r,2r)}\mathrm{gr}_{\mathrm{F}}^{[r+1,2r+2)} \to A$ exists since the composition

$$\mathrm{gr}_{\mathrm{D}}^{[r,2r)}\mathrm{gr}_{\mathrm{F}}^{[r+1,2r+2)} \to \mathrm{gr}_{\mathrm{D}}^{[0,2r)}\mathrm{gr}_{\mathrm{F}}^{[0,2r+2)} \to \mathrm{gr}_{\mathrm{D}}^{[0,r)}\mathrm{gr}_{\mathrm{F}}^{[0,r+1)} \tag{2.3.33}$$

is nullhomotopic. Now using the long homotopy sequence and the commutativity of this diagram we obtain the following diagram:

$$\begin{tikzcd}
& \pi_0\mathrm{gr}_{\mathrm{F}}^{[0,r+1)} \arrow[dl, two heads] \arrow[dd] & \pi_0\mathrm{gr}_{\mathrm{D}}^{[0,\infty)}\mathrm{gr}_{\mathrm{F}}^{[0,r+1)} \arrow[l, "\cong"'] \arrow[r, two heads] \arrow[dd] & \pi_0\mathrm{gr}_{\mathrm{D}}^{[0,r)}\mathrm{gr}_{\mathrm{F}}^{[0,r+1)} \arrow[dd] & \pi_0\mathrm{gr}_{\mathrm{D}}^{[0,r)}\mathrm{gr}_{\mathrm{F}}^{[0,\infty)} \arrow[l, "\cong"'] \arrow[r, two heads] \arrow[dd] & \pi_0\mathrm{gr}_{\mathrm{D}}^{[0,r)}\mathrm{F}^\star \arrow[dr, two heads] \arrow[dd] & \\
E_{r+1}^{0,0}(\mathrm{F}^\star) \arrow[dd, "d"'] & & & & & & E_r^{0,0}(\mathrm{Dec}(\mathrm{F}^\star)) \arrow[dd, "d"] \\
& \pi_{-1}\mathrm{gr}_{\mathrm{F}}^{[r+1,2r+2)} & \pi_{-1}\mathrm{gr}_{\mathrm{D}}^{[0,\infty)}\mathrm{gr}_{\mathrm{F}}^{[r+1,2r+2)} \arrow[l] \arrow[r] & \pi_{-1}A & \pi_{-1}\mathrm{gr}_{\mathrm{D}}^{[r,2r)}\mathrm{gr}_{\mathrm{F}}^{[0,\infty)} \arrow[l, hook', "\beta"'] \arrow[r] & \pi_{-1}\mathrm{gr}_{\mathrm{D}}^{[r,2r)}\mathrm{F}^\star & \\
E_{r+1}^{-r-1,r}(\mathrm{F}^\star) & & & & & & E_r^{-r,r-1}(\mathrm{Dec}(\mathrm{F}^\star)) \\
& \pi_{-1}\mathrm{gr}_{\mathrm{F}}^{[r+1,2r+2)} \arrow[ul, two heads] \arrow[uu, "\mathrm{id}"'] & \pi_{-1}\mathrm{gr}_{\mathrm{D}}^{[r,\infty)}\mathrm{gr}_{\mathrm{F}}^{[r+1,2r+2)} \arrow[l, "\cong"'] \arrow[r, two heads] \arrow[uu, "\alpha", "\cong"'] & \pi_{-1}\mathrm{gr}_{\mathrm{D}}^{[r,2r)}\mathrm{gr}_{\mathrm{F}}^{[r+1,2r+2)} \arrow[uu] & \pi_{-1}\mathrm{gr}_{\mathrm{D}}^{[r,2r)}\mathrm{gr}_{\mathrm{F}}^{[r+1,\infty)} \arrow[l, "\cong"'] \arrow[r, two heads] \arrow[uu] & \pi_{-1}\mathrm{gr}_{\mathrm{D}}^{[r,2r)}\mathrm{F}^\star \arrow[ur, two heads] \arrow[uu, "\mathrm{id}"'] &
\end{tikzcd}$$

where the middle 3×5-diagram and two side-trapezoids commute.

We claim the map α is an isomorphism and β is a monomorphism.

For α, we can argue it as of the map 2 above, the cofiber sequence

$$\mathrm{gr}_{\mathrm{D}}^{[r,\infty)}\mathrm{gr}_{\mathrm{F}}^{[r+1,2r+2)} \to \mathrm{gr}_{\mathrm{D}}^{[0,\infty)}\mathrm{gr}_{\mathrm{F}}^{[r+1,2r+2)} \to \mathrm{gr}_{\mathrm{D}}^{[0,r)}\mathrm{gr}_{\mathrm{F}}^{[r+1,2r+2)} \tag{2.3.34}$$

has a finitely filtered cofiber with graded piece $|\pi_s^{\mathrm{E}}\mathrm{gr}_{\mathrm{F}}^{[r+1,2r+2)}[s]|$ for $0 \leq s < r$. Using Lemma 2.2.6, we see $\pi_a = 0$ for $a \geq -2$ and thus α is an isomorphism.

For β, note that the map fits into the following commutative diagram:

$$\begin{tikzcd}
\pi_0\mathrm{gr}_{\mathrm{D}}^{[0,2r)}\mathrm{gr}_{\mathrm{F}}^{[0,\infty)} \arrow[r] \arrow[d] & \pi_0\mathrm{gr}_{\mathrm{D}}^{[0,r)}\mathrm{gr}_{\mathrm{F}}^{[0,\infty)} \arrow[r] \arrow[d] & \pi_{-1}\mathrm{gr}_{\mathrm{D}}^{[r,2r)}\mathrm{gr}_{\mathrm{F}}^{[0,\infty)} \arrow[r] \arrow[d] & \pi_{-1}\mathrm{gr}_{\mathrm{D}}^{[0,2r)}\mathrm{gr}_{\mathrm{F}}^{[0,\infty)} \arrow[d] \\
\pi_0\mathrm{gr}_{\mathrm{D}}^{[0,2r)}\mathrm{gr}_{\mathrm{F}}^{[0,2r+2)} \arrow[r] & \pi_0\mathrm{gr}_{\mathrm{D}}^{[0,r)}\mathrm{gr}_{\mathrm{F}}^{[0,r+1)} \arrow[r] & \pi_{-1}A \arrow[r] & \pi_{-1}\mathrm{gr}_{\mathrm{D}}^{[0,2r)}\mathrm{gr}_{\mathrm{F}}^{[0,2r+2)}
\end{tikzcd}$$

The same arguments as in 3 show that the left two and the right maps are isomorphisms and this makes β injective.

Now we do some diagram chasing: start from $\pi_0\mathrm{gr}_{\mathrm{D}}^{[0,\infty)}\mathrm{gr}_{\mathrm{F}}^{[0,r+1)}$, the left big trapezoid commutes by assumption and it would be enough to show the following sub-diagram commutes:

$$\begin{tikzcd}
\pi_0\mathrm{gr}_{\mathrm{D}}^{[0,\infty)}\mathrm{gr}_{\mathrm{F}}^{[0,r+1)} \arrow[r, two heads] \arrow[dd] & \pi_0\mathrm{gr}_{\mathrm{D}}^{[0,r)}\mathrm{F}^{\star} \arrow[dr, two heads] & \\
 & & E_r^{0,0}(\mathrm{Dec}(\mathrm{F}^{\star})) \arrow[dd, "d"'] \\
\pi_{-1}\mathrm{gr}_{\mathrm{D}}^{[0,\infty)}\mathrm{gr}_{\mathrm{F}}^{[r+1,2r+2)} & & \\
 & & E_r^{-r,r-1}(\mathrm{Dec}(\mathrm{F}^{\star})) \\
\pi_{-1}\mathrm{gr}_{\mathrm{D}}^{[r,\infty)}\mathrm{gr}_{\mathrm{F}}^{[r+1,2r+2)} \arrow[r, two heads] \arrow[uu, "\cong"'] & \pi_{-1}\mathrm{gr}_{\mathrm{D}}^{[r,2r)}\mathrm{F}^{\star} \arrow[ur, two heads] &
\end{tikzcd}$$

Indeed, this is reduced to show the commutativity of paths from $\pi_0\mathrm{gr}_{\mathrm{D}}^{[0,\infty)}\mathrm{gr}_{\mathrm{F}}^{[0,r+1)}$ to $\pi_{-1}\mathrm{gr}_{\mathrm{D}}^{[r,2r)}\mathrm{gr}_{\mathrm{F}}^{[0,\infty)}$ going top and bottom rows respectively, and this is true by the injectivity of the map β. $\square$

We will return to this isomorphism in Chapter 5.

3 The stable motivic category

In 1980s, when trying to understand slices of algebraic K-theory, Beilinson and Lichtenbaum conjectured the existence of motivic cohomology, aiming to give an analogue of singular cohomology for algebraic varieties. Later in 1990s, Voevodsky and Morel defined their motivic cohomology (of smooth varieties) as the cohomology theory represented by a motivic version of Eilenberg-MacLane spectrum in the stable motivic category $\mathcal{SH}(k)$, where k is a perfect field of characteristic zero.

Generally speaking, motivic homotopy theory is the homotopy theory of smooth schemes where $\mathbb{A}^1$ is the interval object. We first study unstable motivic homotopy theory, whose objects are the so-called motivic spaces, a reasonable analogue of smooth manifolds. Then we will introduce the stable motivic category following the language in [Rob15]. We discuss also the homotopy sheaves of these motivic spectra and pay a special attention to connectivity of them. We fix a quasi-compact quasi-separated base scheme S over a base field k in this chapter.

3.1 Motivic space

Let $\mathcal{S}\mathrm{m}_S$ be the category of smooth schemes of finite type over S. Let $\mathcal{P}(\mathcal{S}\mathrm{m}_S) = \mathrm{Fun}(\mathcal{S}\mathrm{m}_S^{\mathrm{op}}, \mathrm{An})$ be the category of presheaves of animae. Because of Yoneda lemma, the category $\mathcal{S}\mathrm{m}_S$ embeds into it. We shall install a topology on $\mathcal{S}\mathrm{m}_S$.

Definition 3.1.1 The Nisnevich topology on $\mathcal{S}\mathrm{m}_S$ is the Grothendieck topology generated by Nisnevich coverings, i.e. those finite families $\{p_i : U_i \to X\}$ such that each p_i is an étale map and for any field k', the map $\mathrm{Spec}\ k' \to X$ lifts to one of covering maps.

The Nisnevich site has an easier characterization by Nisnevich squares.

Definition 3.1.2 A pullback square in $\mathcal{S}\mathrm{m}_S$

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] & V \arrow[d, "p"'] \\ U \arrow[r, "i"] & X \end{tikzcd}$$

is called an elementary distinguished square, or simply Nisnevich square, if p is an étale map and i is an open immersion, and $p^{-1}(X - i(U)) \to X - i(U)$ is an isomorphism with respect to reduced scheme structures.

We shall mainly focus on sheaves on a Nisnevich site later, therefore it is convenient to have a sheaf condition for a presheaf $\mathcal{F} \in \mathcal{P}(\mathcal{S}\mathrm{m}_S)$.

Definition 3.1.3

1. Let $\mathcal{F}$ be a presheaf on $\mathcal{S}\mathrm{m}_S$, where S is a quasi-compact quasi-separated scheme. $\mathcal{F}$ is said to have Nisnevich excision if $\mathcal{F}(\emptyset) \simeq *$ and for any Nisnevich square $\{U \to X, V \to X\}$ in $\mathcal{S}\mathrm{m}_S$, the induced square

$$\begin{tikzcd} \mathcal{F}(X) \arrow[r] \arrow[d] & \mathcal{F}(U) \arrow[d] \\ \mathcal{F}(V) \arrow[r] & \mathcal{F}(U \times_X V) \end{tikzcd}$$

is a pullback square in An.

2. $\mathcal{F}$ is said to fulfill Nisnevich (Čech) descent if for each Nisnevich covering $\mathcal{U}$ of X, the map

$$\lim \mathrm{Map}_{\mathcal{P}(\mathcal{S}\mathrm{m}_S)}\big(\check{\mathrm{C}}(\mathcal{U}), \mathcal{F}\big) \to \mathcal{F}(X) \tag{3.1.1}$$

is an equivalence.

Theorem 3.1.4 *Let S be a quasi-compact quasi-separated scheme and $\mathcal{F} \in \mathcal{P}(\mathcal{S}\mathrm{m}_S)$, then the followings are equivalent:*

1. $\mathcal{F}$ has Nisnevich excision.

2. $\mathcal{F}$ fulfills Nisnevich descent.

If one of the conditions is fulfilled, then we say $\mathcal{F}$ is a Nisnevich sheaf.

Sketch of proof. We will use the fact that Nisnevich topology over $\mathcal{S}\mathrm{m}_S$ is hypercomplete (cf. [[CM21], Corollary 3.27]), therefore, any Nisnevich descent is also a hyperdescent.

Next, we observe that the Nisnevich squares generate Nisnevich topology, this is [[MV99], §3 ,Proposition 1.4].

Pick a Nisnevich square $\mathcal{U} = \{U \to X, V \to X\}$ and let $c : \check{C}(\mathcal{U}) \to X$ be its Čech nerve. Let $W := U \times_X V$ and X' be the pushout of $U \leftarrow W \to V$ and $k : X' \to X$ be the canonical map.

By definition, $\mathcal{F}$ satisfies Nisnevich excision if and only if $\mathcal{F}(k)$ is an equivalence and $\mathcal{F}(\emptyset) \simeq *$. And by [[AHW17], Lemma 3.1.3] $\mathcal{F}$ satisfies Nisnevich descent if $\mathcal{F}(c)$ is an equivalence and $\mathcal{F}(\emptyset) \simeq *$. If we write

$$\begin{aligned} \mathcal{C} &= \left\{c : \check{C}(\mathcal{U}) \to X\right\} \cup \{e\} \\ \mathcal{K} &= \{k : X' \to X\} \cup \{e\} \end{aligned} \tag{3.1.2}$$

where e is the canonical map from empty sheaf to the sheaf represented by the initial object in $\mathcal{S}\mathrm{m}_S$. Then it suffices to show every map in $\mathcal{C}$ is a $\mathcal{K}$-equivalence and every map in $\mathcal{K}$ is a $\mathcal{C}$ -equivalence. This is shown in [[AHW17], Theorem 3.2.5]. See also [[Lur18], Theorem 3.7.5.1] for a derived algebraic geometry version. □

Write $\mathrm{Shv}(\mathcal{S}\mathrm{m}_S)$ for the ∞-category of Nisnevich sheaves of animae. The unstable motivic category is a certain localization of it.

Definition 3.1.5 The unstable motivic category $\mathcal{H}(S)$ is the localization of $\mathrm{Shv}(\mathcal{S}\mathrm{m}_S)$ under the collection of morphisms $W := \left\{X \times \mathbb{A}^1_S \to X\right\}$ for any $X \in \mathcal{S}\mathrm{m}_S$ in the sense of Definition 6.1.22. We denote $\mathcal{H}(S)_*$ to be the category of minimal pointed motivic spaces as constructed in Remark 6.2.2.

Remark 3.1.6 Thanks to Theorem 3.1.4, we can also exhibit $\mathcal{H}(S)$ as a localization of $\mathcal{P}(\mathcal{S}\mathrm{m}_S)$ under the following morphisms:

1. ($\mathbb{A}^1$-invariance) $X \times \mathbb{A}^1_S \to X$;
2. $よ(U) \coprod_{よ(U \times_X V)} よ(V) \to よ(X)$ for any Nisnevich square $\{U \to X, V \to X\}$;
3. the unique map $\emptyset \to よ(\emptyset)$.

where $よ : \mathcal{S}\mathrm{m}_S \to \mathcal{P}(\mathcal{S}\mathrm{m}_S)$ is the Yoneda embedding functor. This gives us a localization functor $L_{\mathrm{mot}} : \mathcal{P}(\mathcal{S}\mathrm{m}_S) \to \mathcal{H}(S)$.

Theorem 3.1.7 *There is a closed symmetric monoidal structure on $\mathcal{H}(S)_*$ given by the localization of the section-wise smash product.*

Proof. By Proposition 6.4.5 we need to check firstly $\mathcal{H}(S)_*$ has finite products, thus the Cartesian product induces a symmetric monoidal structure on $\mathcal{H}(S)_*$ via the smash product. But it is just the fact that $\mathbb{A}^1$-invariance is preserved by products (and colimits).
Now for closeness. In order to show the product is a left adjoint, we invoke Theorem 6.5.8 (notice $\mathcal{H}(S)_*$ is presentable as an accessible localization) and show it preserves small colimits. Let $l : \mathrm{Shv}(\mathcal{S}\mathrm{m}_S)_* \to \mathcal{H}(S)_*$ be the localization functor with $\iota : \mathcal{H}(S)_* \to \mathrm{Shv}(\mathcal{S}\mathrm{m}_S)_*$ the right adjoint, using $l \circ \iota \simeq \mathrm{id}$ we calculate

$$\begin{aligned}(\mathrm{colim}_i \mathcal{F}_i) \otimes \mathcal{G} &\simeq (\mathrm{colim}_i (l\iota\mathcal{F}_i)) \otimes (l\iota\mathcal{G}) \\ &\simeq l((\mathrm{colim}_i \iota\mathcal{F}_i) \otimes \iota\mathcal{G}) \\ &\simeq l(\mathrm{colim}_i (\iota\mathcal{F}_i \otimes \iota\mathcal{G})) \\ &\simeq \mathrm{colim}_i (l\iota(\mathcal{F}_i \otimes \mathcal{G})) \cong \mathrm{colim}_i (\mathcal{F}_i \otimes \mathcal{G}).\end{aligned} \tag{3.1.3}$$

□

3.2 $\mathbb{P}^1$-invariance and stabilization

Just like motivic spaces resemble the classical homotopy category of spaces, the stable motivic category is a generalization of spectra. In order to define the stabilization of $\mathcal{H}(S)_*$, we need to understand the suspension functor first.

Proposition 3.2.1 *Viewing each space $X \in \mathrm{An}$ as a constant sheaf, the suspension functor in $\mathcal{H}(S)_*$ is given by smash product with $\mathbb{S}^1$.*

Proof. By Theorem 3.1.7 the unit of the symmetric monoidal product is the constant sheaf $\mathbb{S}^0 = * \coprod *$, the definition of suspension functor tells us all. □

By general theory in §6.3, we define:

Definition 3.2.2 The ∞-category $\mathcal{SH}^{\mathcal{S}^1}(S)$ is defined as the stabilization of $\mathcal{H}(S)_*$, i.e. $\mathrm{Sp}(\mathcal{H}(S)_*)$. The objects in this category are called the $\mathcal{S}^1$-motivic spectra.

Instead of using the direct analog $\mathcal{SH}^{\mathcal{S}^1}(S)$ of $\mathcal{SH}$, we would like to invert another important circle in algebraic geometry, namely the Tate circle $\mathbb{G}_m = \mathrm{Spec}\ k[t, t^{-1}]$. We note that $\mathbb{S}^1 \wedge \mathbb{G}_m \simeq \mathbb{P}^1$. Indeed, as $p : \mathbb{A}^1 \to \mathbb{P}^1, x \mapsto (1 : x)$ and $i : \mathbb{A}^1 \to \mathbb{P}^1, x \mapsto (x : 1)$ is a Nisnevich covering of $\mathbb{P}^1$, and there intersection is $\mathbb{G}_m$.

Proposition 3.2.3 *(Voevodsky) Let $\mathcal{C}$ be a symmetric monoidal ∞-category and $X \in \mathcal{C}$. The stabilization $\mathrm{Stab}_X(\mathcal{C})$ is the colimit of the sequence*

$$\mathcal{C} \xrightarrow{X\otimes -} \mathcal{C} \xrightarrow{X\otimes} \mathcal{C} \xrightarrow{X\otimes} \dots \tag{3.2.1}$$

If the action of cyclic permutation on $X \otimes X \otimes X$ becomes an identity map in $\mathrm{Stab}_X(\mathcal{C})$, then $\mathrm{Stab}_X(\mathcal{C})$ has a canonical symmetric monoidal structure and the functor $\mathcal{C} \to \mathrm{Stab}_X(\mathcal{C})$ sending X to an invertible object is monoidal.

Proof: see [[Rob15], Proposition 2.19]. □

Lemma 3.2.4 *In $\mathcal{H}(S)_*$ the 3-cycle map $\sigma : \mathbb{P}^1 \wedge \mathbb{P}^1 \wedge \mathbb{P}^1 \to \mathbb{P}^1 \wedge \mathbb{P}^1 \wedge \mathbb{P}^1$ is homotopic to identity, i.e. $\mathbb{P}^1$ is symmetric in $\mathcal{H}(S)_*$.*

Proof. It suffices to show a transposition on $\mathbb{P}^1 \wedge \mathbb{P}^1$ is homotopic to $-\mathrm{id}$, then we can use the fact that a 3-cycle in S_3 is a composition of 2 transpositions.
Consider the $\mathrm{SL}_2(\mathbb{Z})$ action on $\mathbb{P}^1 \wedge \mathbb{P}^1$ and we see the two matrices are $\mathbb{A}^1$-homotopic:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \simeq \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.2.2}$$

since they can be related with elementary transformations. Indeed, consider the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.2.3}$$

and we have

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.2.4}$$

Thus for a transposition τ we have: $\tau \simeq -\mathrm{id} \wedge \mathrm{id} \simeq -\mathrm{id}$. □

Definition 3.2.5 The stable motivic category $\mathcal{SH}(S)$ is the colimit of the following sequence:

$$\mathcal{H}(S)_* \xrightarrow{\mathbb{P}^1 \wedge -} \mathcal{H}(S)_* \xrightarrow{\mathbb{P}^1 \wedge -} \mathcal{H}(S)_* \xrightarrow{\mathbb{P}^1 \wedge -} \dots \tag{3.2.5}$$

together with a symmetric monoidal functor $\Sigma^\infty_{\mathbb{P}^1} : \mathcal{H}(S)_* \to \mathcal{SH}(S)$ which sends $\mathbb{P}^1$ to an invertible object. Moreover, $\mathcal{SH}(S)$ carries a canonical symmetric monoidal structure.

Remark 3.2.6 As $\mathcal{H}(S)_*$ is presentable, the functor $\Sigma^\infty_{\mathbb{P}^1}$ preserves all small colimits, so by Theorem 6.5.8 there is a right adjoint $\Omega^\infty_{\mathbb{P}^1}$ which preserves all small limits.

In order to give a reasonable definition of "inverting $\mathbb{P}^1$", we need to use the construction given in [[Rob15], §2.1], which not only proves the stability of resulting category but is also accompanied with a nice universal property to work with.
For every small symmetric monoidal ∞-category $\mathcal{C}$, we shall denote $\mathrm{CAlg}_{\mathcal{C}}$ the category of small symmetric monoidal ∞-categories $\mathcal{D}$ with a monoidal structure map $\mathcal{C} \to \mathcal{D}$. We have a full subcategory spanned by those $\mathcal{D}$'s whose structure map sends $X \in \mathcal{C}$ to an invertible

object, written as $\mathrm{CAlg}_{\mathcal{C}}^{X}$. By [[Rob15], Proposition 2.1], there is a left adjoint of the inclusion $\mathrm{CAlg}_{\mathcal{C}}^{X} \hookrightarrow \mathrm{CAlg}_{\mathcal{C}}$, written as $\mathcal{L}_{\mathcal{C},X}$

Definition 3.2.7 [[Rob15], Definition 2.6] Let $\mathcal{C}$ be a presentable symmetric monoidal ∞-category and let $X \in \mathcal{C}$. The formal inversion of X in $\mathcal{C}$ is the presentable symmetric monoidal ∞-category $\mathcal{C}[X^{-1}]$ defined by the pushout

$$\mathcal{C}[X^{-1}] = \mathcal{C} \coprod_{\mathcal{P}(\mathrm{free}^{\otimes}(\Delta[0]))} \mathcal{P}\Big(\mathcal{L}_{\mathrm{free}^{\otimes}(\Delta[0]),*}(\mathrm{free}^{\otimes}(\Delta[0]))\Big) \tag{3.2.6}$$

in $\mathrm{CAlg}(\mathrm{Pr}^L)$, where $\mathrm{free}^{\otimes}(\Delta[0])$ is the free symmetric monoidal category generated by $*$ and we interpret the monoidal map $\mathrm{free}^{\otimes}(\Delta[0]) \to \mathcal{C}$ to be the object $X \in \mathcal{C}$.

Note by [[Rob15], Proposition 2.9], the formal inversion is canonically equivalent to the category $\mathcal{L}^{\mathrm{Pr}}_{\mathcal{C},X}(\mathcal{C})$, where $\mathcal{L}^{\mathrm{Pr}}_{\mathcal{C},X}$ is the restriction of $\mathcal{L}_{\mathcal{C},X}$ onto $\mathrm{CAlg}(\mathrm{Pr}^L)$, making it initial among all presentable symmetric monoidal ∞-categories such that X is invertible. This suggests an equivalence of this formal inversion with the more familiar notation of stabilization.
Since the tensor product in Pr^L preserves all small colimits by [[Lur17], Proposition 4.8.1.17], the canonical map $\mathcal{C} \to \mathcal{C}[X^{-1}]$ produces a forgetful functor $\mathrm{Mod}_{\mathcal{C}[X^{-1}]}(\mathrm{Pr}^L) \to \mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^L)$ and the base change functor $\mathcal{L}^{\mathrm{Pr}}_{\mathcal{C},X} := (- \otimes_{\mathcal{C}} \mathcal{C}[X^{-1}])$. This base change functor is monoidal, left adjoint to $\mathrm{CAlg}_{\mathcal{C}[X^{-1}]}(\mathrm{Pr}^L) \to \mathrm{CAlg}_{\mathcal{C}}(\mathrm{Pr}^L)$. We have an adjunction pair

$$F : \mathrm{CAlg}_{\mathcal{C}[X^{-1}]}(\mathrm{Pr}^L) \leftrightarrows \mathrm{CAlg}_{\mathcal{C}}(\mathrm{Pr}^L) : \mathcal{L}^{\mathrm{Pr}}_{\mathcal{C},X}. \tag{3.2.7}$$

Theorem 3.2.8 *[[Rob15], Corollary 2.22] Let $\mathcal{C}$ be a presentable symmetric monoidal ∞-category and let X be a symmetric object in $\mathcal{C}$. The map of $\mathcal{C}$-modules*

$$\mathcal{L}^{\mathrm{Pr}}_{\mathcal{C},X}(M) \to \mathcal{L}^{\mathrm{Pr}}_{\mathcal{C},X}(\mathrm{Stab}_X(M)) \to \mathrm{Stab}_X(M) \tag{3.2.8}$$

induced by adjunction (3.2.7) is an equivalence. In particular, the (underlying ∞-category of) formal inversion $\mathcal{C}[X^{-1}]$ is equivalent to $\mathrm{Stab}_X(\mathcal{C})$.

Corollary 3.2.9 *The stable motivic category $\mathcal{SH}(S)$ is indeed stable.*

Remark 3.2.10 There is also an adjunction pair of suspension and desuspension between $\mathcal{SH}^{\mathbb{S}^1}(S)$ and $\mathcal{SH}(S)$. In fact, by using the same technique as in Definition 3.2.7 we see that $\mathcal{SH}(S) \simeq \mathcal{SH}^{\mathbb{S}^1}(S)[\mathbb{G}_m^{-1}]$, see for example [[ABH24], 2.2.3].

Finally we have the following universal property of $\mathcal{SH}(S)$:

Proposition 3.2.11 *[[Rob15], Corollary 2.39] Let $\mathcal{C}$ be a pointed presentable symmetric monoidal ∞-category, the composition with stabilization*

$$\mathrm{Fun}^{\otimes,L}(\mathcal{SH}(S), \mathcal{C}) \to \mathrm{Fun}^{\otimes}(\mathcal{H}(S)_*, \mathcal{C}), F \mapsto F \circ \Sigma^{\infty}_{\mathbb{P}^1} \tag{3.2.9}$$

is fully faithful with essential image consisting of those symmetric monoidal functors $F : \mathcal{H}(S)_* \to \mathcal{C}$ which are $\mathbb{P}^1$-stable, i.e. the homotopy cofiber of $F(S) \to F(\mathbb{P}^1)$ induced from $S \xrightarrow{\infty} \mathbb{P}^1$ is $\otimes$-invertible.

Remark 3.2.12 Many arithmetic cohomology theories turn out to be not $\mathbb{A}^1$-invariant, therefore, the condition $\mathbb{A}^1$-invariance in motivic homotopy theory is too strong for this context. In fact, if we take the same formal inversion procedure on the ∞-topos $\mathcal{S}\mathrm{t}_*$, the resulted ∞-category is a reasonable generalization of motivic spectra as by [AHI25]. We also note that in this case, the first construction as in Proposition 3.2.3 does not work any more, hence computations are more difficult.

The analogue of homotopy groups of spectra are homotopy sheaves.

Definition 3.2.13 Let $E \in \mathcal{SH}(S)$, we shall denote $\mathbb{S}^{i,j} := (\mathbb{S}^1)^{\wedge i-j} \wedge \mathbb{G}_m^{\wedge j}$ to be the motivic (i,j) -sphere. The (i,j)-th homotopy sheaf $\pi_{i,j}(E)$ of E is the sheafification of the presheaf

$$X \in \mathcal{S}\mathrm{m}_S \mapsto [\Sigma^\infty_{\mathbb{P}^1} X_+, E \wedge \mathbb{S}^{-i,-j}]_{\mathcal{SH}(S)} \tag{3.2.10}$$

where $[X, Y]$ is the 0-th truncation of the mapping space, i.e. the set of morphisms in the homotopy category.

Definition 3.2.14 For $E, F \in \mathcal{SH}(S)$ and $X \in \mathcal{S}\mathrm{m}_S$, we define

1. the E-cohomology of X as

$$E^{p,q}(X) := [\Sigma^\infty_{\mathbb{P}^1} X_+, E \wedge \mathbb{S}^{p,q}]_{\mathcal{SH}(S)} \tag{3.2.11}$$

2. and the E-cohomology of F as

$$E^{p,q}(F) := [F, E \wedge \mathbb{S}^{p,q}]_{\mathcal{SH}(S)}. \tag{3.2.12}$$

3. Dually we can define the E-homology of F as

$$E_{p,q}(F) := [E \wedge \mathbb{S}^{-p,-q}, F]_{\mathcal{SH}(S)}. \tag{3.2.13}$$

3.3 $\mathbb{A}^1$-connectivity theorem

Morel's stable $\mathbb{A}^1$-connectivity theorem [[Mor05], Theorem 6.1.8] serves as a useful tool for determining homotopy sheaves of a spectrum. It is later essentially used for showing equivalences of effective covers and slices of spectra, for example, Corollary 4.4.2.

Morel established the theorem over a perfect base field. This was later generalized under a series of papers, to a Noetherian scheme of finite dimension by [Dru22] and to a qcqs scheme of finite valuative dimension by [BK25].

For our purpose we only need the theorem for a base field k. We then give a proof here following the strategy in [Ayo21] and [BM23], after some preliminary definitions. In this section we fix a base field k.

Definition 3.3.1 Let τ be a Grothendieck topology on $\mathcal{S}\mathrm{m}_k$, let $\mathcal{F} \in \mathcal{P}(\mathcal{S}\mathrm{m}_k)_*$ be a pointed presheaf. $\mathcal{F}$ is said to be locally n-connected (with respect to τ) if the τ-sheafification of $\pi_i\mathcal{F}$, i.e., of $U \mapsto \pi_j\mathcal{F}(U)$ is trivial for any $j \leq n$.

Unless otherwise stated, locally n-connected always means locally n-connected with respect to the Nisnevich topology. We state the main theorem of this section.

Theorem 3.3.2

1. *(Unstable $\mathbb{A}^1$-connectivity) Suppose $\mathcal{F} \in \mathcal{P}(\mathcal{S}\mathrm{m}_k)_*$ is locally n-connected, then so is the motivic localization of $\mathcal{F}$ in $\mathcal{H}(k)_*$.*
2. *(Stable $\mathbb{A}^1$-connectivity) Suppose $\mathcal{F} \in \mathcal{P}(\mathcal{S}\mathrm{m}_k)_*$ is locally n-connected, then so is the motivic spectrum $\Sigma^\infty_{\mathbb{P}^1}(L_{\mathrm{mot}}\mathcal{F}) \in \mathcal{SH}(k)$.*

Notice the stable connectivity theorem follows from the unstable one immediately by how we defined the homotopy sheaf of a spectrum in Definition 3.2.13 (see also [[Dru22], §2.1]). Therefore, we only prove the unstable one.

There are two technical definitions of connectivity that are used in the proof.

Definition 3.3.3 Let $\mathcal{F} \in \mathcal{P}(\mathcal{S}\mathrm{m}_k)_*$ be a pointed presheaf.

1. $\mathcal{F}$ is said to be generically n-connected if for any connected $X \in \mathcal{S}\mathrm{m}_k$ with generic point η_X, $\pi_j\mathcal{F}(\eta_X) = 0$ for any $j \leq n$.
2. $\mathcal{F}$ is said to be n-preconnected if for any local essentially smooth scheme X over k with dimension $\dim X = m$, $\pi_j\mathcal{F}(X^h) = 0$ for any $j \leq n - m$, where X^h is the henselization of X.

If $\mathcal{F}$ is locally n-connected, then $\mathcal{F}$ is n-preconnected. And the definition of preconnectedness is a Nisnevich local condition, therefore a presheaf is n-preconnected if and only if its Nisnevich sheafification is.

Lemma 3.3.4 *Let $\mathcal{F} \in \mathrm{Shv}(\mathcal{S}\mathrm{m}_k)_*$ be a Nisnevich sheaf. If $\mathcal{F}$ is n-preconnected, then for any essentially smooth k-scheme X of dimension d, $\pi_j\mathcal{F}(X) = 0$ for any $j \leq n - d$.*

Proof. For every point $x \in X$, we have $\dim \overline{\{x\}} \leq d - \dim(X_x)$. The statement now follows from [[CM21], Theorem 3.30]. □

Theorem 3.3.5 *[[Ayo21], Théorème 4.12] If $F \in \mathcal{P}(\mathcal{S}\mathrm{m}_k)_*$ is n-preconnected, then so is its motivic localization.*

Proof. Write $\mathrm{Sing}^{\mathbb{A}^1}(\mathcal{F}) := |\mathcal{F}(- \times \Delta^\bullet)|$ the singular construction as in [[MV99], §2.3], then by [[AE16], Theorem 4.27], the motivic localization L_{mot} is the countable iteration $\left(a_{\mathrm{Nis}}\mathrm{Sing}^{\mathbb{A}^1}\right)^{\circ\mathbb{N}}$. It suffices to check $\mathrm{Sing}^{\mathbb{A}^1}$ preserves preconnectedness. Now for a Nisnevich sheaf $\mathcal{F}$ and X an essentially smooth scheme of dimension d, Lemma 3.3.4 implies that $\pi_j\mathcal{F}(X \times \Delta^m) = 0$ for any

$j \leq n - m - d$. Now we can conclude using the Bousfield-Kan spectral sequence associated to the the tower

$$\ldots \to \mathcal{F}(X \times \Delta^m) \to \mathcal{F}(X \times \Delta^{m+1}) \to \ldots \tag{3.3.1}$$

$\square$

Proof of Theorem 3.3.2. As $\mathcal{F}$ is locally n-connected, $\mathcal{F}$ is n-preconnected. Theorem 3.3.5 indicates that the motivic localization $L_{\mathrm{mot}}\mathcal{F}$ is also n-preconnected. Since preconnectedness implies generically connectedness (a generic point has always dimension 0), we know $L_{\mathrm{mot}}\mathcal{F}$ is generically n-connected.

Now we can check the vanishing condition Zariski-locally for $x \in X$. Inspect the definition, we can assume X is local, connected with generic point η_X. We use the Bloch-Ogus-Gabber theorem for effaceablity [[CHK97], Theorem 5.1.10], this gives us

$$\ker(\pi_i(L_{\mathrm{mot}}\mathcal{F}(X)) \to \pi_i(L_{\mathrm{mot}}\mathcal{F}(\eta_X))) = 0 \tag{3.3.2}$$

for any $i \leq n$. Indeed, the assumption SUB1 and SUB2 for $L_{\mathrm{mot}}\mathcal{F}$ before the statement of this theorem is fulfilled, as we can see that the commutative diagram

$$\begin{tikzcd} \mathbb{A}^1_X \arrow[r, hook, "j_X"] \arrow[dr, "\pi_X"'] & \mathbb{P}^1_X \arrow[d] & X \arrow[l, hook', "\infty"'] \arrow[dl, equal] \\ & X & \end{tikzcd}$$

induces an $\mathbb{A}^1$-homotopy of maps j_X^* and $(\infty \circ \pi_X)^*$ from $L_{\mathrm{mot}}\mathcal{F}(\mathbb{P}^1_X)$ to $L_{\mathrm{mot}}\mathcal{F}(\mathbb{A}^1_X)$.

Now, since $L_{\mathrm{mot}}\mathcal{F}$ is generically n-connected, we have $\pi_i(L_{\mathrm{mot}}\mathcal{F}(\eta_X)) = 0$ for $i \leq n$, hence $\pi_i\big(L_{\mathrm{mot}}\mathcal{F}(\mathrm{Spec}\ \mathcal{O}_{X,x})\big) = 0$ for all x, this proves the theorem. $\square$

As an application, we can define a notion of connectivity of motivic spectra, and show this defines a t-structure on $\mathcal{SH}(S)$.

Definition 3.3.6 The homotopy t-structure on $\mathcal{SH}(S)$ is given by

$$\begin{aligned} \mathcal{SH}(S)_{\geq 0} &= \big\{E \in \mathcal{SH}(S) : \pi_{i,j}(E) = 0, \forall i - j < 0\big\} \\ \mathcal{SH}(S)_{\leq 0} &= \big\{E \in \mathcal{SH}(S) : \pi_{i,j}(E) = 0, \forall i - j > 0\big\}. \end{aligned} \tag{3.3.3}$$

Theorem 3.3.7 *The homotopy t-structure is indeed a t-structure. And all truncation functors are symmetric monoidal with respect to the smash product on $\mathcal{SH}(S)$.*

Proof. Write $\pi_i(E)_j := \pi_{i-j,j}(E)$, the definition above for $\mathcal{SH}(S)$ can be rewritten as $\pi_i(E)_* = 0, \forall i < 0$, which corresponds to the smashing with $(\mathbb{S}^1)^{\wedge i}$. Using the adjunction in Remark 3.2.10 we see that $\pi_i(E)_* = \pi_i\big(\Omega^\infty_{\mathbb{G}_m}(E \wedge \mathbb{S}^{*,*})\big)$. Therefore, it suffices to show that

$$\mathcal{SH}^{\mathcal{S}^1}(S)_{\geq 0} = \left\{E \in \mathcal{SH}^{\mathcal{S}^1}(S) : \pi_i(E) = 0, \forall i < 0\right\}$$
$$\mathcal{SH}^{\mathcal{S}^1}(S)_{\leq 0} = \left\{E \in \mathcal{SH}^{\mathcal{S}^1}(S) : \pi_i(E) = 0, \forall i > 0\right\} \tag{3.3.4}$$

defines a t-structure on $\mathcal{SH}^{\mathcal{S}^1}(S)$. Now let E be a $\mathcal{S}^1$-spectrum and $E_{\geq 0}$ be the non-negative part of E induced by the Postnikov truncation on $\mathcal{SH}$, therefore $\pi_i(E_{\geq 0}) = 0, \forall i < 0$. By Theorem 3.3.2 $L_{\text{mot}}(E_{\geq 0})$ is also -1-connectedsl. Hence we have a split exact sequence in $\mathcal{P}(\mathcal{S}\text{m}_S, \mathcal{SH})$:

$$L_{\text{mot}}(E_{\geq 0}) \to E_{\geq 0} \to E. \tag{3.3.5}$$

Since L_{mot} preserves finite colimits, we see that $E_{\geq 0}$ is also in $\mathcal{SH}^{\mathcal{S}^1}(S)$.
We define the localization functor

$$L : \mathcal{SH}^{\mathcal{S}^1}(S) \to \mathcal{SH}^{\mathcal{S}^1}(S)$$
$$E \mapsto E_{\geq 0}. \tag{3.3.6}$$

and since its essential image is closed under extension, by Proposition 6.2.23 it is a t-localization and induces a t-structure. □

3.4 Some motivic spectra

The first important spectrum is the sphere spectrum, which plays the role of unit of the symmetric monoidal structure on $\mathcal{SH}(S)^{\otimes}$.

Definition 3.4.1 The motivic sphere spectrum $\mathbb{S}_S \in \mathcal{SH}(S)$ is given by $\Sigma^{\infty}_{\mathbb{P}^1}\mathbb{S}^{0,0}$, i.e. the $\mathbb{P}^1$ stabilization of the constant sheaf $\mathbb{S}^{0,0}$.

The stable motivic category allows us to define cohomology theories on (smooth) varieties. We shall consider some of them, while concentrate on the geometric nature.
One could ask whether there is an analogue of the Eilenberg-MacLane spectrum $H_{\mathbb{Z}} \in \mathcal{SH}$ in the motivic world, whereas

$$\pi_i H_{\mathbb{Z}}(n) = \begin{cases} \mathbb{Z} \text{ if } i = n \\ 0 \text{ if } i \neq n \end{cases} \tag{3.4.1}$$

However, since the homotopy sheaf of a motivic spectrum is bigraded, it is impossible to directly construct a similar spectrum. Therefore we will define the motivic Eilenberg-MacLane spectrum $M_{\mathbb{Z}}$ as the spectrum representing Voevodsky's motivic cohomology in [Voe98].

Construction 3.4.2 Let R be a regular k-algebra for k a field and $S = \text{Spec}(R)$. Let $\mathbb{Z}_{\text{tr}} : \mathcal{S}\text{m}_S \to \text{Fun}(\mathcal{S}\text{m}_S^{\text{op}}, \text{Ab})$ be the functor such that for $X, U \in \mathcal{S}\text{m}_S$, $\mathbb{Z}_{\text{tr}}(X)(U)$ is the free abelian group generated by finite correspondences from U to X, i.e. by those closed irreducible subsets of $U \times X$ which are finite and surjective over a connected component of U.
Let $C_\bullet\mathbb{Z}_{\text{tr}}(X)$ be the simplicial presheaf $U \mapsto \mathbb{Z}_{\text{tr}}(X)(U \times \Delta^\bullet)$ and $C_*\mathbb{Z}_{\text{tr}}(X)$ be the associated chain complex of presheaves. By Dold-Kan correspondence, $C_\bullet\mathbb{Z}_{\text{tr}}(X)$ is quasi-isomorphic to

$C_*\mathbb{Z}_{\mathrm{tr}}(X)$. By [[MVW06], Corollary 2.24] this chain complex is $\mathbb{A}^1$-invariant. By [[MVW06], Lemma 6.2] $\mathbb{Z}_{\mathrm{tr}}(X)$ is an étale sheaf and therefore also a Nisnevich sheaf. We therefore get a functor $L : \mathcal{S}\mathrm{m}_S \to \mathcal{H}(S)$ by sending X to $C_\bullet\mathbb{Z}_{\mathrm{tr}}(X)$.

Since for every morphism $f : U \to X$, the graph $\Gamma(f)$ is a finite correspondence, we get a map $\Gamma(X) : X \to L(X)$ by Yoneda embedding. In this case, the exterior product on spaces induces the smash product in $\mathcal{H}(S)_*$ and we have:

$$\begin{aligned} \Sigma_{\mathbb{S}^1} : \mathbb{S}^1 \wedge L(\mathbb{S}^{i,j}) &\xrightarrow{\Gamma(\mathbb{S}^1)\wedge \mathrm{id}} L(\mathbb{S}^1) \wedge L(\mathbb{S}^{i,j}) \to L(\mathbb{S}^{i+1,j}) \\ \Sigma_{\mathbb{G}_m} : \mathbb{G}_m \wedge L(\mathbb{S}^{i,j}) &\xrightarrow{\Gamma(\mathbb{G}_m)\wedge \mathrm{id}} L(\mathbb{G}_m) \wedge L(\mathbb{S}^{i,j}) \to L(\mathbb{S}^{i+1,j+1}) \end{aligned} \tag{3.4.2}$$

Definition 3.4.3

1. The motivic Eilenberg-MacLane spectrum $M_\mathbb{Z}$ is $\Sigma^\infty_{\mathbb{P}^1} L(\mathbb{S}^{2,1}) \in \mathcal{SH}(S)$. It can be seen as a $(\mathbb{S}^1, \mathbb{G}_m)$-bispectrum whose (i,j)-th degree is given by $L(\mathbb{S}^{i+j,j})$.
2. The motivic cohomology of a spectrum $E \in \mathcal{SH}(S)$ is given by

$$H^{p,q}(E) := [E, \mathbb{S}^{p,q} \wedge M_\mathbb{Z}]_{\mathcal{SH}(S)} \tag{3.4.3}$$

Remark 3.4.4 Bloch first gave a possible construction of motivic cohomology of smooth varieties over a field k using a complex built from algebraic cycles in [Blo86]. Voevodsky proposed Construction 3.4.2 in the context of motivic homotopy theory and showed it is related to Bloch's cycle complex as in [Voe04] and [Lev08].

Spitzweck constructed a $\mathcal{E}_\infty$-ring spectrum $M_\mathbb{Z} \in \mathcal{SH}(\mathbb{Z})$ as in [[Spi18], Definition 4.27]. For any integral scheme S with $f : S \to \operatorname{Spec} \mathbb{Z}$, the $\mathcal{E}_\infty$-algebra $M_S := f^*(M_\mathbb{Z}) \in \mathcal{SH}(S)$ becomes a module over $M_\mathbb{Z}$ and is a well-behaved $\mathbb{A}^1$-motivic cohomology theory.

One of the key features of $M_\mathbb{Z}$ is that it is the zeroth slice of the slice filtrations (s. §4.1) on KGL, the spectrum representing algebraic K-theory defined later. This is the approach used in [BEM25] to define an $\mathbb{A}^1$-motivic cohomology over any qcqs scheme, using Weibel's homotopy K-theory KH instead of K. On regular schemes, this recovers Voevodsky's construction.

In [Bou24] a motivic cohomology theory over any qcqs scheme was constructed combining [BEM25] and trace methods. This theory is in general not $\mathbb{A}^1$-invariant, and after $\mathbb{A}^1$-localization this matches the construction in [BEM25], [Spi18] and [Voe98] as shown in [[Bou24], §6].

The following spectrum plays an important role in this thesis.

Let S be a qcqs scheme. As usual, for a vector bundle $\mathcal{E} \to X$ over smooth scheme $X \in \mathcal{S}\mathrm{m}_S$, we can define the Thom space $\mathrm{Th}(\mathcal{E}) := \mathcal{E} \ / \ (\mathcal{E} - X) \in \mathrm{Shv}(\mathcal{S}\mathrm{m}_S)_*$ pointed at the image of $\mathcal{E} - X$, where X is embedded as the zero section. Suppose we have $\mathcal{E}_1 \to X_1$ and $\mathcal{E}_2 \to X_2$, then

$$\mathrm{Th}(\mathcal{E}_1 \times \mathcal{E}_2 \to X_1 \times X_2) = \mathrm{Th}(\mathcal{E}_1) \wedge \mathrm{Th}(\mathcal{E}_2). \tag{3.4.4}$$

Definition 3.4.5 Let

$$\mathrm{BGL}_n = \mathrm{Gr}_n := \lim_k \mathrm{Gr}_n\left(\mathbb{A}^{n+k}\right) \tag{3.4.5}$$

be the Grassmannian of n-dimensional affine subspaces and γ_n the tautological bundle on it. The product $\mathbb{A}^1 \times \gamma_n \to \mathrm{BGL}_n$ is classified by pullback of the canonical map $\mathrm{BGL}_n \to \mathrm{BGL}_{n+1}$ and γ_{n+1}, this induces a bundle map $\mathbb{A}^1 \times \gamma_n \to \gamma_{n+1}$, and by (3.4.4) a structure map

$$\mathrm{Th}(\mathbb{A}^1) \wedge \mathrm{Th}(\gamma_n) \simeq \mathbb{P}^1 \wedge \mathrm{Th}(\gamma_n) \to \mathrm{Th}(\gamma_{n+1}). \tag{3.4.6}$$

The algebraic cobordism spectrum $\mathrm{MGL}_S \in \mathcal{SH}(S)$ is defined to be

$$\mathrm{MGL}_S := \mathrm{colim}_n \Sigma^{-n}_{\mathbb{P}^1} \Sigma^{\infty}_{\mathbb{P}^1} \mathrm{Th}(\gamma_n). \tag{3.4.7}$$

Definition 3.4.6 Let S be a regular scheme. Using the same notation as in Definition 3.4.5, we can define the algebraic K-theory spectrum $\mathrm{KGL}_S \in \mathcal{SH}(S)$ as

$$\mathrm{KGL}_S := \Sigma^{\infty}_{\mathbb{P}^1} L_{\mathrm{mot}}(\mathbb{Z} \times \mathrm{BGL}) \tag{3.4.8}$$

where BGL is the sequential limit of

$$\ldots \hookrightarrow \mathrm{BGL}_n \hookrightarrow \mathrm{BGL}_{n+1} \hookrightarrow \ldots \tag{3.4.9}$$

and $L_{\mathrm{mot}}(\mathbb{Z} \times \mathrm{BGL})$ the motivic localization of $\mathbb{Z} \times \mathrm{BGL}$ as in Remark 3.1.6.
The structure map is given by

$$\beta : \mathbb{P}^1 \wedge L_{\mathrm{mot}}(\mathbb{Z} \times \mathrm{BGL}) \to L_{\mathrm{mot}}(\mathbb{Z} \times \mathrm{BGL}) \tag{3.4.10}$$

representing the Bott element in $K^0(\mathbb{P}^1 \wedge L_{\mathrm{mot}}(\mathbb{Z} \times \mathrm{BGL}))$.

Remark 3.4.7 The $\mathcal{E}_\infty$-ring structure of MGL (and KGL) will be handled later systematically via the construction of motivic Thom spectra. However, there's also a base independent $\mathcal{E}_\infty$-ring structure on KGL as proven in [NSØ15].

We introduce the notion of oriented spectra in $\mathcal{SH}(S)$ and show that MGL is actually universal among those. We first state an important theorem regarding normal bundles: the homotopy purity theorem.

Theorem 3.4.8 *[[MV99], §3, Theorem 2.23] Suppose there is a closed embedding of smooth schemes $i : Z \hookrightarrow X$. Let $N_{X,Z}$ be the normal bundle of Z in X, then*

$$\Sigma^{\infty}_{\mathbb{P}^1} \mathrm{Th}\left(N_{X,Z}\right) \simeq \Sigma^{\infty}_{\mathbb{P}^1}(X \;/\; X - i(Z)) \tag{3.4.11}$$

Definition 3.4.9 Let $(E, \mu, 1_E)$ be a ring spectrum in $\mathcal{SH}(S)$ and $i : \left(\mathbb{P}^1_S, \infty\right) \to (\mathbb{P}^\infty_S, \infty)$ be the inclusion as pointed spaces. An orientation on E is a class $c \in E^{2,1}(\mathbb{P}^\infty_S)$, i.e., the $(2,1)$-th E -cohomology of $\mathbb{P}^1$, such that $i^*(c) = 1_E$, here we view $E^{2,1}\left(\mathbb{P}^1_S\right) = E^{0,0}(\mathbb{S}_S)$. A pair (E, c) is called an oriented spectrum.

Note that MGL_S is canonically oriented by the class

$$x : \Sigma^\infty_{\mathbb{P}^1} \mathbb{P}^\infty_S \simeq \Sigma^\infty_{\mathbb{P}^1} \mathrm{Th}(\gamma_1) \to \mathrm{MGL}_S \wedge \mathbb{P}^1_S \tag{3.4.12}$$

since the restriction of this map on $(\mathbb{P}^1_S, \infty)$ is

$$\Sigma^\infty_{\mathbb{P}^1} \mathbb{S}^{0,0} = \Sigma^\infty_{\mathbb{P}^1} \mathrm{Th}(\gamma_0) \to \mathrm{MGL}_S \tag{3.4.13}$$

by homotopy purity theorem. Moreover, this orientation is universal as we can see from the following theorem.

Remark 3.4.10 By similar arguments KGL is also canonically oriented. But there is also an algebraic counterpart of real topological K-theory, just like KGL being the analogue of complex topological K-theory. (s. Theorem 4.4.7) This is the so called Hermitian K-theory spectrum KO as constructed in [Hor05] together with a motivic real Bott periodicity. Unlike KGL, KO is not oriented.

Theorem 3.4.11 *[[NSØ09], Proposition 6.2 & Corollary 6.7] Let $E \in \mathcal{SH}(S)$ be a ring spectrum, there is a bijection of ring spectra maps $\varphi : \mathrm{MGL}_S \to E$ and orientations on E, given by $\varphi \mapsto \varphi^*(x^{\mathrm{MGL}})$.*

One last motivic spectrum needed to state our main theorems is the motivic Brown-Peterson spectrum constructed in [Vez01].
Fix a prime ℓ and let $\mathrm{MGL}_{(\ell)}$ be the Bousfield localization of MGL at ℓ and $L : \mathrm{MGL} \to \mathrm{MGL}_{(\ell)}$ the localization map. We have an induced isomorphism

$$L^* : \mathrm{MGL}^{*,*}_{(\ell)}(\mathrm{MGL}_{(\ell)}) \to \mathrm{MGL}^{*,*}_{(\ell)}(\mathrm{MGL}) \tag{3.4.14}$$

by [[Lur10], Lecture 20, Example 4]. Let x^{MGL} be the canonical orientation of MGL, then $\mathrm{MGL}_{(\ell)}$ is oriented by $L(x) =: x_{(\ell)}$. As of [[Rud98], VII.6.2], we have an associated formal group law $F_{x_{(\ell)}}$ on $\mathrm{MGL}^*_{(\ell)} := \oplus_i \mathrm{MGL}^{2i,i}_{(\ell)}$.
Recall that a homomorphism between formal group laws F and G is a power series $f(x)$ such that $f(F(x,y)) = G(f(x), f(y))$. If the coefficient of x is invertible, we say f is an isomorphism, and if the coefficient of x is 1, we say f is a strict isomorphism. We call a strict isomorphism from F to the additive formal group law $x + y$ the logarithm of F.

Definition 3.4.12 Let ℓ be a prime. A formal group law F over a $\mathbb{Z}_{(\ell)}$-algebra is called ℓ-typical if its logarithm takes the form $\sum_{i \geq 0} a_i x^{\ell^i}$ with $a_0 = 1$.

In order to have a direct analog of topological Brown-Peterson spectrum in chromatic homotopy theory. We want to have a universal ℓ-typical formal group law (see also the discussion before Corollary 4.2.7). We use the following theorem due to Cartier:

Theorem 3.4.13 *[[Haz12], 16.4.14] Let A be a $\mathbb{Z}_{(\ell)}$-algebra, then every formal group law over A is strictly isomorphic to a ℓ-typical group law over A.*

Set $A = \mathrm{MGL}^*_{(\ell)}$ in the above theorem, we can assume $F_{x_{(\ell)}}$ ℓ-typical. By Theorem 3.4.11 this is related to a map

$$e : \mathrm{MGL} \to \mathrm{MGL}_{(\ell)}. \tag{3.4.15}$$

The unique ring spectra map

$$e_{(\ell)} : \mathrm{MGL}_{(\ell)} \to \mathrm{MGL}_{(\ell)} \tag{3.4.16}$$

such that $L^*\left(e_{(\ell)}\right) = e$ is called the motivic Quillen idempotent. Since $e_{(\ell)}$ is associated to an ℓ-typical formal group law, it is idempotent by [[Haz12], 31.1.9].

Definition 3.4.14 The motivic Brown-Peterson spectrum $\mathrm{BP}^{(\ell)}_{\mathrm{mot}}$ at a prime ℓ is defined as the sequential colimit of

$$\ldots \xrightarrow{e_{(\ell)}} \mathrm{MGL}_{(\ell)} \xrightarrow{e_{(\ell)}} \mathrm{MGL}_{(\ell)} \xrightarrow{e_{(\ell)}} \ldots \tag{3.4.17}$$

In particular, $\mathrm{BP}^{(\ell)}_{\mathrm{mot}}$ is a direct summand of $\mathrm{MGL}_{(\ell)}$.

As shown later in §4.2, we can define the motivic Brown-Peterson spectrum directly from motivic Landweber theory proven in [NSØ09], assuming the Hopkins-Morel-Hoyois isomorphism. However, the discussion here provides an explicit construction.

3.5 Thom spectra

In this section we want to establish an $\mathcal{E}_\infty$-ring structure on MGL over any base scheme S. The result is crucial later in the proof of our main theorem Theorem 5.2.1. A priori, Definition 3.4.5 already gives us a hint on constructing ring structure using the induced map $\gamma_n \times \gamma_m \to \gamma_{n+m}$. But this approach is computationally overwhelmed and does not directly give us an element in $\mathrm{CAlg}(\mathcal{SH}(S))$. Instead, we follow the approach in [[BH21], §16] and define a motivic counterpart of the Thom spectrum functor.

Let $\mathcal{SH} : \mathcal{S}\mathrm{m}_S^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^L)$ be the stable functor constructed in Definition 3.2.7, note this functor is by construction a spherical presheaf in the sense of Proposition 6.5.13.

We write $\mathrm{Span} := \mathrm{Span}(\mathcal{S}\mathrm{m}_S, \mathrm{all}, \mathrm{fold})$ as the category of spans defined in Definition 6.6.1. In view of Proposition 6.6.4, $\mathcal{SH}$ is a product-preserving functor $\mathcal{SH} : \mathrm{Span} \to \mathrm{Pr}^L$. In fact, there are more good properties of this functor:

In order to build multiplicative structures, we need to borrow some concepts from the six-functor formalism. This was mainly done by Ayoub in [Ayo07].

Proposition 3.5.1 *[[Ayo07], §1.4.1]*

1. *For any $X, Y \in \mathcal{S}\mathrm{m}_S$ we have $\mathcal{SH}(X \coprod Y) \simeq \mathcal{SH}(X) \times \mathcal{SH}(Y)$, thus a fold map $\nabla : Y \to Z$ induces a functor $\nabla_\otimes : \mathcal{SH}(Y) \to \mathcal{SH}(Z)$ by smash product.*
2. *For any $f : Y \to X$ smooth in $\mathcal{S}\mathrm{m}_S$, we have two symmetric monoidal (with respect to smash product) adjunction pairs*

$$
\begin{aligned}
f^* : \mathcal{SH}(Y) &\leftrightarrows \mathcal{SH}(X) : f_* \\
f_\sharp : \mathcal{SH}(X) &\leftrightarrows \mathcal{SH}(Y) : f^*.
\end{aligned} \tag{3.5.1}
$$

And $f_\sharp, f^*$ satisfy the projection formula

$$
f_\sharp(A \wedge f^* B) \simeq f_\sharp(A) \wedge B. \tag{3.5.2}
$$

3. *For any Cartesian square*

$$
\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "g"] & Y
\end{tikzcd}
$$

where f and f' are smooth we have smooth base change, i.e.

$$
\mathrm{Ex}^*_\sharp : f'_\sharp(g')^* \Longrightarrow g^* f_\sharp \tag{3.5.3}
$$

is an equivalence.

4. *[[BH21], Proposition 5.10] for u, u' smooth and ∇, ∇' fold maps in the commutative diagram*

$$
\begin{tikzcd}
W \arrow[d, "g"'] & Y' \arrow[l, "f'"'] \arrow[r, "\nabla'"] \arrow[d, "u'"] & Z' \arrow[d, "u"', no head] \\
X & Y \arrow[l, "f"] \arrow[r, "\nabla"] & Z
\end{tikzcd}
$$

there is a distributivity transformation

$$
\mathrm{Dist}_{\sharp,\otimes} : u_\sharp \nabla'_\otimes (f')^* \Longrightarrow \nabla_\otimes (\pi_Y)_\sharp (\pi_W)^* \tag{3.5.4}
$$

and this is an equivalence if the right square is a pullback and $Z' = R_{Y/Z}(W \times_X Y)$ is a Weil restriction in the sense of [[BH21], §2.3].

Proposition 3.5.2 *For any $X \in \mathcal{S}\mathrm{m}_S$, $\mathcal{SH}(X)$ admits sifted colimits and $\mathcal{SH}$ lifts to a functor in* $\mathrm{Cat}_\infty^{\mathrm{sift}}$.

Proof. Note first for any X, $\mathcal{SH}(X)$ is cocomplete and thus admits sifted colimit. In order to lift $\mathcal{SH}$, we need to show that for any $X \stackrel{f}{\leftarrow} Y \stackrel{\nabla}{\rightarrow} Z$, where f is smooth and ∇ is a fold map, the induced map $\nabla_\otimes \circ f^* : \mathcal{SH}(X) \to \mathcal{SH}(Z)$ as in Proposition 3.5.1 preserves sifted colimits. As a left adjoint, f^* preserves all colimits and since $\nabla_\otimes$ is induced from the smash product, it commutes with sifted colimits. □

Theorem 3.5.3 *Let $\mathcal{L}$ be the collection of smooth quasi-projective morphisms in* Sch_S. *There is a strict natural transformation*

$$
M : \mathcal{P}_\Sigma\left(\mathcal{L}_{/\mathcal{SH}}\right) \to \mathcal{SH} \tag{3.5.5}
$$

extending

$$\mathcal{L}_{/\mathcal{SH}} \to \mathcal{SH}, f \mapsto f_\sharp \tag{3.5.6}$$

by objectwise sifted cocompletion and M preserves colimits.

Proof. We give the proof in amount of [[Ban25], Theorem 2.20]. Let $\mathrm{Fun}_{\mathcal{L}}(\Delta^1, \mathrm{Span}) \subset \mathrm{Fun}(\Delta^1, \mathrm{Span})$ be the full subcategory generated by spans $X \xleftarrow{f} Y \xrightarrow{\nabla} Z$ with $f \in \mathcal{L}$. Let $s, t : \mathrm{Fun}(\Delta^1, \mathrm{Span})$ be source and target functors. The composition of $\mathcal{SH}$ with evaluation $\mathrm{ev} : \mathrm{Fun}_{\mathcal{L}}(\Delta^1, \mathrm{Span}) \times \Delta^1 \to \mathrm{Span}$ yields a natural transformation

$$\varphi : \mathcal{SH} \circ s \to \mathcal{SH} \circ t : \mathrm{Fun}_{\mathcal{L}}(\Delta^1, \mathrm{Span}) \to \mathrm{Cat}_\infty^{\mathrm{sift}}. \tag{3.5.7}$$

Suppose $E : \mathcal{E} \to \mathrm{Span}^{\mathrm{op}}$ is a Cartesian fibration classified by $\mathcal{SH}$, then φ can be viewed as a map $\varphi : s^*\mathcal{E} \to t^*\mathcal{E}$ in $\mathrm{Fun}_{\mathcal{L}}(\Delta^1, \mathrm{Span})^{\mathrm{op}}$ as [[Lur09], Definition 3.3.2.2]. For a smooth map $f : Y \to X$ in $\mathcal{L}$, the fiber of φ over f is the pullback $f^* : \mathcal{SH}(X) \to \mathcal{SH}(Y)$. By Proposition 3.5.1, it has a left adjoint $f_\sharp$. By [[BH21], Lemma D.3], the fiberwise adjoint of φ gives a relative adjunction

$$\psi : t^*\mathcal{E} \leftrightarrows s^*\mathcal{E} : \varphi. \tag{3.5.8}$$

On components, ψ encodes the map $f_\sharp$ by construction, therefore the naturality follows from the smooth base change $\mathrm{Ex}_\sharp^*$. We consider the following diagram

$$\begin{tikzcd}
t^*\mathcal{E} \arrow[r, "\psi"] \arrow[dr, "t^*E"] & s^*\mathcal{E} \arrow[r, "\chi"] \arrow[d, "s^*E"'] & \mathcal{E} \arrow[d, "E"'] \\
 & \mathrm{Fun}_{\mathcal{L}}(\Delta^1, \mathrm{Span})^{\mathrm{op}} \arrow[r, "s^{\mathrm{op}}"] & \mathrm{Span}^{\mathrm{op}}
\end{tikzcd}$$

where the square is Cartesian by construction. As a Weil restriction [[BH21], §2.3], s is a coCartesian fibration, and therefore $s^{\mathrm{op}} \circ t^*E$ is a Cartesian fibration, and so does $\chi \circ \varphi$, its fiber over $X \in \mathrm{Sch}_S$ is a functor

$$(\mathcal{L}_X)_{/\!\!/\mathcal{SH}} \to \mathcal{SH}(X), (f : Y \to X, P \in \mathcal{SH}(Y)) \mapsto f_\sharp P \tag{3.5.9}$$

where $(\mathcal{L}_X)_{/\!\!/\mathcal{SH}}$ is the Cartesian fibration classified by $\mathcal{SH}$. It remains to check $\chi \circ \psi$ preserves Cartesian edges.

We inspect the behavior of $s^{\mathrm{op}} \circ t^*E$ in detail. Let e be an $(s^{\mathrm{op}} \circ t^*E)$-Cartesian edge, then $t^*E(e)$ is the opposite of an edge in $\mathrm{Fun}_{\mathcal{L}}(\Delta^1, \mathrm{Span})$ of the form

$$\begin{tikzcd}
W \arrow[d, "\mathrm{id}"'] & Y' \arrow[l] \arrow[r] \arrow[d, "\mathrm{id}"'] & Z' \arrow[d, "\mathrm{id}"'] \\
W \arrow[d, "g"'] & Y' \arrow[l, "f'"'] \arrow[r, "\nabla'"] \arrow[d, "u'"] & Z' \arrow[d, "u"] \\
X & Y \arrow[l, "f"'] \arrow[r, "\nabla"] & Z
\end{tikzcd}$$

where the down right square is a pullback square and $Y' = R_{Y\,/\,Z}(W \times_X Y) \times_Z Y$ and $Z' = R_{Y\,/\,Z}(W \times_X Y)$. Since e is a Cartesian lift of $t^*(E)(e)$, it is of the form

$$\alpha : \left(Z \leftarrow Z' \xrightarrow{\mathrm{id}} Z', H\right) \to \left(X \leftarrow W \xrightarrow{\mathrm{id}} W, F\right) \tag{3.5.10}$$

with $H \in \mathcal{SH}(Z')$ and $F \in \mathcal{SH}(W)$, where α consists in the data in the diagram above together with an equivalence $H \simeq \nabla'_{\otimes}(f')^* F$.

By commutativity $\chi(\psi(e))$ is an edge $\left(Z, u_\sharp H\right) \to \left(X, g_\sharp F\right)$ is given by $(X \leftarrow Y \to Z)^{\mathrm{op}}$ and $u_\sharp H \to \nabla_{\otimes} f^* g_\sharp F$ as the composition of

$$u_\sharp H \xrightarrow{\simeq} u_\sharp \nabla'_{\otimes}(f')^* F \xrightarrow{\mathrm{Dist}_{\sharp,\otimes}} \nabla_{\otimes}(\pi_Y)_\sharp(\pi_W)^* F \xrightarrow{\mathrm{Ex}^*_\sharp} \nabla_{\otimes} f^* g_\sharp F \tag{3.5.11}$$

where $\pi_Y : Y \times_X W \to Y$ and $\pi_W : Y \times_X W \to W$ are projections. By the last two statements of Proposition 3.5.1 we see this is indeed an equivalence, thus $\chi(\psi(e))$ is E-Cartesian.

Now we can restrict the functor $\chi \circ \psi$ to the wide subcategory $\mathcal{L}_{/\mathcal{SH}}$ in $\mathcal{L}_{/\!\!/\mathcal{SH}}$. Finally Proposition 3.5.2 together with Proposition 6.5.13 gives a lift

$$M : \mathcal{P}_\Sigma\left(\mathcal{L}_{/\mathcal{SH}}\right) \to \mathcal{SH} \tag{3.5.12}$$

of $\chi \circ \psi$ by objectwise sifted cocompletion. $\square$

Definition 3.5.4 The colimit-preserving functor $M_S : \mathcal{P}_\Sigma\left((\mathcal{S}\mathrm{m}_S)_{/\mathcal{SH}}\right) \to \mathcal{SH}(S)$ is called the motivic Thom spectrum functor.

Remark 3.5.5

1. Notice that $\mathcal{P}_\Sigma\left((\mathcal{S}\mathrm{m}_S)_{/\mathcal{SH}}\right) \simeq \mathcal{P}_\Sigma(\mathcal{S}\mathrm{m}_S)_{/\mathcal{SH}} \simeq \mathcal{P}_\Sigma(\mathcal{S}\mathrm{m}_S)_{/\mathcal{SH}^{\simeq}}$ by [[ABG18], §5.3], where $\mathcal{SH}^{\simeq}$ is the functor sending a scheme S to the core of $\mathcal{SH}(S)$ as defined in Definition 6.1.15.
2. Let $X \in \mathcal{P}_\Sigma(\mathcal{S}\mathrm{m}_S)$ and $\varphi : X \to \mathcal{SH}$ be a natural transformation. Since X is a colimit of representable objects, we have $\varphi \simeq \mathrm{colim}_{\alpha : U \to X} \varphi \circ \alpha$ and since M_S is colimit-preserving and the extension of $(\mathcal{S}\mathrm{m}_S)_{/\mathcal{SH}}$, we have
$$M_S(\varphi) \simeq \mathrm{colim}_{f : U \to S, \alpha \in X(U)} f_\sharp \varphi_U(\alpha). \tag{3.5.13}$$
3. [[BH21], Proposition 16.9] The Thom spectrum functor M_S automatically satisfies Nisnevich descents, but is not always $\mathbb{A}^1$-invariant. On the other hand, it is plausible that the restriction of M_S onto $\mathcal{P}(\mathcal{S}\mathrm{m}_S)_{/\mathrm{Sph}}$ is $\mathbb{A}^1$-invariant. This is currently unknown.

Let $\mathrm{Vect}(X)$ be the symmetric monoidal ∞-category of vector bundles over $X \in \mathcal{S}\mathrm{m}_S$, with symmetric monoidal structure given by direct sums. We have a symmetric monoidal functor

$$\mathrm{Vect}(X) \to \mathrm{Shv}_{\mathrm{Nis}}(\mathcal{S}\mathrm{m}_X), \xi \mapsto \mathrm{Th}(\xi) \tag{3.5.14}$$

and this is natural in X. As localization and stabilization are symmetric monoidal, we obtain

$$\mathrm{Vect}(X) \to \mathrm{Sph}(X), \xi \mapsto \Sigma^\infty_{\mathbb{P}^1}\mathrm{Th}(\xi) \tag{3.5.15}$$

still natural in X, where $\mathrm{Sph}(X)$ is the anima spanned by invertible objects in $\mathcal{SH}(X)^{\simeq}$. And this gives us a natural transformation

$$\mathrm{Vect} \to \mathrm{Sph} : \mathrm{Span} \to \mathrm{CAlg}(\mathrm{An}). \tag{3.5.16}$$

Since $\mathrm{Sph}(X)$ is an anima, by taking the group completion of $\mathrm{Vect}(X)$, we have the factorization

$$\begin{tikzcd} \mathrm{Vect} \arrow[r] \arrow[d] & \mathrm{Sph} \\ \mathrm{Vect}^{\mathrm{gp}} \arrow[r] \arrow[ur, dashed] & K \arrow[u, dashed, "j"'] \end{tikzcd}$$

by Proposition 6.6.6, since K is a right Kan extension of $\mathrm{Vect}^{\mathrm{gp}}$ and as a Zariski sheaf, Sph is the right Kan extension of its restriction onto affine spaces.

Definition 3.5.6 The above natural transformation $j : K \to \mathrm{Sph}$ is called the motivic J-homomorphism.

Proposition 3.5.7 *Let $e : K^{\circ} \hookrightarrow K$ be the injection of rank 0 part of algebraic K-theory. The bundle γ : $\mathrm{BGL} \to K^{\circ}$ representing $\mathbb{A}^1$-localization of tautological bundle induces an equivalence in $\mathcal{SH}(S)$:*

$$\mathrm{MGL}_S = M_S(j \circ e \circ \gamma) \simeq M_S(j \circ e). \tag{3.5.17}$$

Proof. We will still focus the case when S is a regular scheme. The general case also follows from smooth base change to Spec $\mathbb{Z}$.

By Remark 3.5.5 and the fact that K is an $\mathbb{A}^1$-invariant sheaf on regular schemes (see e.g. [[TT90], Theorem 10.8]), $M_S(j \circ -)$ inverts $\mathbb{A}^1$-homotopy, satisfies Nisnevich descents and has the explicit representation

$$M_S(j \circ e \circ \gamma) \simeq \mathrm{colim}_{(X,\gamma_X)} (p_X)_\sharp (\Sigma^\infty_{\mathbb{P}^1} \mathrm{Th}(\gamma_X)) = \mathrm{MGL}_S. \tag{3.5.18}$$

It remains to show γ is already $\mathbb{A}^1$-invariant, for this, we consider the commutative diagram

$$\begin{tikzcd} \mathrm{BGL} \arrow[r] \arrow[d, "f"'] & K_{\circ} \arrow[d, "g"'] \\ \mathrm{B_{\acute{e}t}GL} \arrow[r] & K^{\circ} \end{tikzcd}$$

where $K_{\circ}$ the connected component of 0 in K° and $\mathrm{B_{\acute{e}t}GL}$ the étale classifying space of GL. f is a motivic equivalence by [[MV99], §4, Proposition 2.6] and g is a Zariski equivalence. Therefore, it suffices to compare $L_{\mathbb{A}^1}\mathrm{BGL} \to L_{\mathbb{A}^1}K_{\circ}$ on affine covers. But it is an equivalence as a homological equivalence between connected H-space. $\square$

Using this equivalence we can equip a commutative algebra structure on MGL_S.

Corollary 3.5.8 *The algebraic cobordism spectrum is equipped with an $\mathcal{E}_\infty$-ring structure.*

Proof. Since M_S is colimit preserving, it sends objects in $\mathrm{CAlg}\Big(\mathcal{P}_\Sigma(\mathcal{S}\mathrm{m}_S)_{/\mathcal{SH}^{\simeq}}\Big)$ to objects in $\mathrm{CAlg}(\mathcal{SH}(S))$. It remains to check that the motivic J-homomorphism on zeroth summand is a commutative algebra object in $\mathcal{P}_\Sigma(\mathcal{S}\mathrm{m}_S)_{/\mathcal{SH}^{\simeq}}$. We observe this follows from the following lemma, which is purely algebraic and has nothing to do with our main purpose. □

Lemma 3.5.9 *[[BH21], Proposition 16.17] There is a functor*

$$M|_{\mathcal{L}} : \mathcal{P}_\Sigma(\mathrm{Span}^{\mathrm{op}})_{/\mathcal{SH}} \to \mathrm{CAlg}\Big(\mathcal{P}_\Sigma(\mathcal{S}\mathrm{m}_S)_{/\mathcal{SH}}\Big) \tag{3.5.19}$$

induced by the target functor $t : \mathrm{Fun}_{\mathcal{L}}(\Delta^1, \mathrm{Span}) \to \mathrm{Span}$, *where* $\mathrm{Fun}_{\mathcal{L}}(\Delta^1, \mathrm{Span})$ *is the full subcategory generated by* $X \xleftarrow{f} Y = Y$ *for* f *a smooth quasi-projective morphism.*

Remark 3.5.10 This $\mathcal{E}_\infty$ structure is different from the one defined in [GS09] by inverting the Bott element in every stage of BGL. In fact, [HY20] shows that the two structure must differ after complex Betti realization (see also Theorem 4.4.1) and is only an isomorphism of $\mathcal{E}_2$-rings, supposing S has complex points.

4 Motivic filtrations and realizations

4.1 Motivic slice tower

To give a reasonable justification of what connectivity in $\mathcal{SH}(S)$ means, one can try to define the notion of effective spectra as in [Voe02].

Definition 4.1.1

1. The full subcategory of $\mathcal{SH}(S)$ generated by spectra $\Sigma^n_{\mathbb{S}^1}\Sigma^\infty_{\mathbb{P}^1}X_+, n \in \mathbb{Z}$ under colimits, where $X \in \mathcal{S}\mathrm{m}_S$ a smooth scheme, is called the category of effective motivic spectra, denoted by $\mathcal{SH}^{\mathrm{eff}}(S)$.
2. For any $k \in \mathbb{Z}$, the category of k-effective spectra $\mathcal{SH}^{\mathrm{eff}}(S)(k)$ is generated by

$$\Sigma^n_{\mathbb{S}^1}\Sigma^\infty_{\mathbb{P}^1}\Big((\mathbb{P}^1)^{\wedge k} \wedge X\Big)_+, n \in \mathbb{Z}. \tag{4.1.1}$$

   where $X \in \mathcal{S}\mathrm{m}_S$ a smooth scheme.

Remark 4.1.2

1. $\mathcal{SH}^{\mathrm{eff}}(S)$ is a stable ∞-category, since by definition it contains all colimits and the desuspension $\Omega \simeq \Sigma^{-1}_{\mathbb{S}^1}$ of any object.
2. The category of effective spectra does not contain any $\mathbb{P}^1$-desuspension of $\Sigma^\infty_{\mathbb{P}^1}X_+$, this can be rephrased into certain connectivity with respect to $\mathbb{G}_m$, as we have $\mathbb{P}^1 \simeq \mathbb{S}^1 \wedge \mathbb{G}_m$.
3. We have a natural tower of full embeddings:

$$\ldots\mathcal{SH}^{\mathrm{eff}}(S)(k+1) \subset \mathcal{SH}^{\mathrm{eff}}(S)(k) \subset \mathcal{SH}^{\mathrm{eff}}(S)(k-1) \subset \ldots \tag{4.1.2}$$

Definition 4.1.3 On $\mathcal{SH}^{\mathrm{eff}}(S)$ we define the effective homotopy t-structure as follows:

$$\begin{aligned} \mathcal{SH}^{\mathrm{eff}}(S)_{\geq 0} &:= \mathcal{SH}^{\mathrm{eff}}(S) \cap \mathcal{SH}(S)_{\geq 0} \\ \mathcal{SH}^{\mathrm{eff}}(S)_{\leq 0} &:= \mathcal{SH}^{\mathrm{eff}}(S) \cap \mathcal{SH}(S)_{\leq 0} \end{aligned} \tag{4.1.3}$$

where $(\mathcal{SH}(S)_{\geq 0}, \mathcal{SH}(S)_{\leq 0})$ is the homotopy t-structure defined in Definition 3.3.6.

Proposition 4.1.4 *The inclusion functor $\iota_k : \mathcal{SH}^{\mathrm{eff}}(S)(k) \to \mathcal{SH}(S)$ admits a right adjoint r_k : $\mathcal{SH}(S) \to \mathcal{SH}^{\mathrm{eff}}(S)(k)$.*

Proof. This is again an example of adjoint functor theorem Theorem 6.5.8. It suffices to check $\mathcal{SH}^{\mathrm{eff}}(S)(k)$ is presentable and ι_k preserves colimits.

By definition, $\mathcal{SH}^{\mathrm{eff}}(S)(k)$ is generated under colimits by a small set of objects $\Sigma^{2k+m,k}\Sigma^{\infty}_{\mathbb{P}^1}X_+$. Now it's clear that X_+ is compact in $\mathcal{P}(\mathcal{S}\mathrm{m}_S)$ and the inclusion $\mathcal{H}(S)_* \to \mathcal{P}(\mathcal{S}\mathrm{m}_S)_*$ preserves filtered colimits. By construction Ω^∞ preserves filtered colimits and ι_k preserves colimits since $\mathcal{SH}^{\mathrm{eff}}(S)(k)$ is closed under all small colimits. We can conclude using Lemma 6.5.6 and Theorem 6.5.7. $\square$

For a motivic spectrum E, we set $E^k := \iota_k(r_k(E))$, called the k-effective cover of E. We then have a filtration

$$\ldots \to E^{k+1} \to E^k \to E^{k-1} \to \ldots \tag{4.1.4}$$

Definition 4.1.5 The filtration (4.1.4) is called the slice filtration of motivic spectra. The graded piece is called the slice.

This filtration is exhaustive: to see this, notice $\iota_k \circ r_k$ preserves colimits and for every effective spectrum $\Sigma^{\infty}_{\mathbb{P}^1}X_+$, there exists a $k \in \mathbb{Z}$ such that $\Sigma^{\infty}_{\mathbb{P}^1}X_+ \in \mathcal{SH}^{\mathrm{eff}}(S)(k)$. Since $\iota_k r_k \circ \iota_n r_n \simeq \iota_k r_k$ for $n < k$, we conclude that

$$\begin{aligned} \mathrm{Map}_{\mathcal{SH}(S)}(\Sigma^{\infty}_{\mathbb{P}^1}X_+, \iota_k r_k \mathrm{colim}_n E^n) &\simeq \mathrm{Map}_{\mathcal{SH}^{\mathrm{eff}}(S)(k)}(\Sigma^{\infty}_{\mathbb{P}^1}X_+, r_k(E)) \\ &\simeq \mathrm{Map}_{\mathcal{SJ}(S)}(\Sigma^{\infty}_{\mathbb{P}^1}X_+, E). \end{aligned} \tag{4.1.5}$$

This filtration is not always complete. This will be related to the convergence of slice spectral sequences later in §5.1.

Remark 4.1.6 The primary aim of Voevodsky to introduce slice filtration on spectra is to relate motivic cohomology with algebraic K-theory by a spectral sequence (see [[Voe02], Conjecture 7, §7]), which mimics the spectral sequence of singular cohomology that converges to topological K-theory of a space. This was established over a Dedekind domain in [Lev08] and over arbitrary regular base scheme in [BEM25].

Voevodsky's notion of slices is not compatible with the homotopy t-structures on $\mathcal{SH}(S)$ and $\mathcal{SH}$. For example, suppose we want our realization functor (defined in §4.3) to be t-exact, one

has to make sure that the desuspension functor Σ^{-n} or $\Sigma^{-n,0}$ changes connectedness in both $\mathcal{SH}$ and $\mathcal{SH}(S)$, however, this is not the case in the latter one.
Instead, we consider a notion of very effective spectra as introduced in [SØ12].

Definition 4.1.7

1. The full subcategory in $\mathcal{SH}(S)$ generated under colimits by spectra $\Sigma^n_{\mathbb{S}^1}\Sigma^\infty_{\mathbb{P}^1}X_+, n \geq 0$ is called the category of very effective spectra, denoted by $\mathcal{SH}^{\text{veff}}(S)$.
2. For any $k \in \mathbb{Z}$, the category of k-very effective spectra is generated by
$$\Sigma^n_{\mathbb{S}^1}\Sigma^\infty_{\mathbb{P}^1}\left((\mathbb{P}^1)^{\wedge k} \wedge X\right)_+, n \geq 0. \tag{4.1.6}$$
where $X \in \mathcal{S}\mathrm{m}_S$ a smooth scheme.

Example 4.1.8 [[BH21], Lemma 13.1] Let X be a smooth scheme and ξ be a bundle of rank n over X. Let $\Sigma^\infty_{\mathbb{P}^1}\mathrm{Th}(\xi)$ be the Thom spectrum of ξ as constructed in §3.5, then $\Sigma^\infty_{\mathbb{P}^1}\mathrm{Th}(\xi)$ is n-very effective. In particular, MGL is very effective.

The follow proposition justifies the relation between effective and very effective spectra.

Proposition 4.1.9 *[[Bac17], Proposition 4]*

1. *We have $\mathcal{SH}^{\text{veff}}(S) = \mathcal{SH}^{\text{eff}}(S)_{\geq 0}$.*
2. *The functor $r_0 : \mathcal{SH}(S) \to \mathcal{SH}^{\text{eff}}(S)$ is t-exact.*

Remark 4.1.10

1. Unlike the category of effective spectra, the subcategory of very effective spectra is not stable as a truncation with respect to the t-structure.
2. This subcategory is presentable by Proposition 6.5.11. And by Proposition 6.2.23, the effective homotopy t-structure is the uniquely accessible t-structure on $\mathcal{SH}^{\text{eff}}(S)$ determined by the collection of objects $\Sigma^\infty_{\mathbb{P}^1}X_+$ where $X \in \mathcal{S}\mathrm{m}_S$.

By the same argument in Proposition 4.1.4, for the inclusion $\tilde{\imath}_k : \mathcal{SH}^{\text{veff}}(S)(k) \to \mathcal{SH}(S)$ we have a right adjoint $\tilde{r}_k : \mathcal{SH}(S) \to \mathcal{SH}^{\text{veff}}(S)(k)$, and this induces a tower of very effective cover of spectra:
$$\ldots \to \tilde{E}^{k+1} := \tilde{\imath}_{k+1}\tilde{r}_{k+1}E \to \tilde{E}^k \to \tilde{E}^{k-1} \to \ldots \tag{4.1.7}$$

Definition 4.1.11 The filtration (4.1.7) is called the generalized slice filtration of motivic spectra. The graded piece is called the generalized slice.

We collect some interesting (generalized) slices of motivic spectra in the following example.

Examples 4.1.12

1. [[Voe04], Theorem 6.6] If S is essentially smooth over a field, then $\mathrm{gr}^0\mathbb{S}^*_S \simeq M_{\mathbb{Z}}$.

2. [[Lev08], Theorem 9.0.3] If S is the spectrum of a Dedekind domain, then $\mathrm{gr}^0\mathbb{S}^*_S \simeq M_{\mathbb{Z}}$. An argument for any qcqs scheme can be found in [BEM25].
3. [[Lev08], Theorem 6.4.2, Theorem 9.0.3] The slices of KGL^* over a perfect field are all isomorphic to the zero slice of $\mathbb{S}_k$. Again the case for qcqs schemes is treated in [BEM25].
4. [[Bac17], Theorem 16] The generalized slices of $\widetilde{\mathrm{KO}}^*$ are given by

$$\tilde{\mathrm{gr}}^n\widetilde{\mathrm{KO}}^* \simeq \Sigma^n_{\mathbb{P}^1} \wedge \begin{cases} \tilde{\mathrm{gr}}^0\widetilde{\mathrm{KO}}^* & n \equiv 0 (\mathrm{mod}\, 4) \\ M_{\mathbb{Z}} \,/\, 2 & n \equiv 1 (\mathrm{mod}\, 4) \\ M_{\mathbb{Z}} & n \equiv 2 (\mathrm{mod}\, 4) \\ 0 & n \equiv 3 (\mathrm{mod}\, 4) \end{cases} \tag{4.1.8}$$

where $\tilde{\mathrm{gr}}^0\widetilde{\mathrm{KO}}^*$ fits into two decompositions

$$\begin{gathered} M_{\mathbb{Z}} \,/\, 2[1] \to \tilde{\mathrm{gr}}^0\widetilde{\mathrm{KO}}^* \to \widetilde{M}_{\mathbb{Z}} \\ \mathbb{G}_m \wedge M^W_{\mathbb{Z}} \to \tilde{\mathrm{gr}}^0\widetilde{\mathrm{KO}}^* \to M_{\mathbb{Z}} \end{gathered} \tag{4.1.9}$$

here $\widetilde{M}_{\mathbb{Z}} := \tilde{\iota}_0\tilde{r}_0 K^{\mathrm{MW}}_*$ is the generalized motivic cohomology and $M^W_{\mathbb{Z}}$ is the Witt-motivic cohomology. This result matches the topological nature of KO, i.e. the Bott periodicity.

4.2 Hopkins-Morel isomorphism

We are now going to determine the zeroth slice of algebraic cobordism. This is an ingredient in our proof of main theorem and has fundamental importance on transferring the theory of chromatic homotopy theory into algebraic setting.

In this section we fix k a field of exponential characteristic e, that is, if $\mathrm{char}\; k = 0$, then $e = 1$ else $e = \mathrm{char}\; k$. Unless otherwise stated, the connectedness of a spectrum always means connectivity with respect to homotopy t-structure of $\mathcal{SH}(k)$ in Definition 3.3.6.

For any oriented ring spectrum $E \in \mathcal{SH}(k)$, we have

$$E^{*,*}\mathrm{BGL}_r \cong E^{*,*}[[c_1, \ldots, c_n]] \tag{4.2.1}$$

where c_i is the i-th Chern class of the tautological bundle. Just like in topology, this isomorphism comes from the calculation of $E^{*,*}(\mathrm{Gr}_r(\mathbb{A}^{k+r}))$ and a limit argument [[NSØ09], Proposition 6.2 (i)]. For $\beta_n \in E_{*,*}\mathrm{BGL}$, the element dual to c_1^n, an easy calculation shows that

$$E_{*,*}\mathrm{BGL} \cong E_{*,*}[\beta_1, \beta_2, \ldots]. \tag{4.2.2}$$

Since the restriction $E^{*,*}\mathrm{BGL} \to E^{*,*}\mathbb{P}^\infty$ kills all higher Chern classes, we know β_n's span $E_{*,*}\mathbb{P}^\infty$.

Now under the Thom isomorphism [[NSØ09], Proposition 6.2 (iii)]

$$E^{*,*}\mathbb{P}^\infty \cong \tilde{E}^{*,*}\Sigma^{-2,-1}\mathrm{MGL}_1 \tag{4.2.3}$$

we have dual elements $b_n \in \tilde{E}_{2n,n}\Sigma^{-2,-1}\mathrm{MGL}_1$ and the isomorphism

$$E_{*,*}\mathrm{MGL} \simeq E_{*,*}\mathrm{BGL} \simeq E_{*,*}[b_1, b_2, \ldots]. \tag{4.2.4}$$

If we take $E = M_{\mathbb{Z}}$, from this isomorphism we get a map

$$\mathrm{MGL}/(b_1, b_2, \ldots) \to M_{\mathbb{Z}}. \tag{4.2.5}$$

Now we can formulate the Hopkins-Morel-Hoyois isomorphism [[Hoy15], Theorem 7.12].

Theorem 4.2.1 *(Hopkins-Morel-Hoyois) Let k be a field of exponential characteristic e and $\mathrm{MGL} \in \mathcal{SH}(k)$ the algebraic cobordism spectrum. The canonical map*

$$f : \mathrm{MGL}/(b_1, b_2, \ldots)[1/e] \to M_{\mathbb{Z}}[1/e] \tag{4.2.6}$$

is an equivalence.

Remark 4.2.2 Theorem 4.2.1 is an direct analogue of Quillen's theorem, saying

$$\mathrm{MU}/(b_1, b_2, \ldots) \simeq H\mathbb{Z}. \tag{4.2.7}$$

Put $L := \mathbb{Z}[b_1, b_2, \ldots]$ the Lazard ring, the theorem above can be rephrased as

$$L[1/e] \cong \mathrm{MGL}_{2,1}(k)[1/e] \tag{4.2.8}$$

classifying all formal group laws [[Hoy15], Proposition 8.2].

We write Λ for $\mathrm{MGL}/(b_1, b_2, \ldots)$ for simplicity.

Lemma 4.2.3 *$M_{\mathbb{Z}} \wedge f : M_{\mathbb{Z}} \wedge \Lambda[1/e] \to M_{\mathbb{Z}} \wedge M_{\mathbb{Z}}[1/e]$ is an equivalence.*

Sketch of proof. As noted in [Hoy15], it is reduced to check $H_{\mathbb{Q}} \wedge f$ and $H_{\mathbb{Z}/l} \wedge f$ are equivalences. Indeed, set $F := \mathrm{fib}(f)$, we need to show $M_{\mathbb{Z}} \wedge F = 0$. As $\mathcal{SH}(k)$ is a presentable ∞-category, we check on compact object $X \in \mathcal{SH}(k)$ that $[X, M_{\mathbb{Z}} \wedge F]_{\mathcal{SH}(k)} = 0$. It suffices to check

$$\begin{gathered}
[X, M_{\mathbb{Z}} \wedge F] \otimes_{\mathbb{Z}} \mathbb{Q} = 0 \\
[X, M_{\mathbb{Z}} \wedge F] \otimes_{\mathbb{Z}} \mathbb{Z}/l = 0 \\
\mathrm{Tor}^1([X, M_{\mathbb{Z}} \wedge F], \mathbb{Z}/l) = 0.
\end{gathered} \tag{4.2.9}$$

As is shown in [[Hoy15], Proposition 4.13], the motivic Eilenberg-MacLane construction $H : \mathrm{Sp}(\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Ab})) \to \mathcal{SH}(S)$, where $\mathrm{Sp}(\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Ab}))$ is the category of spectrum objects in simplicial abelian groups, is colimit preserving, as X compact, we have

$$[X, M_{\mathbb{Z}} \wedge F] \otimes_{\mathbb{Z}} \mathbb{Q} = \left[X, H_{\mathbb{Q}} \wedge F\right] = 0 \tag{4.2.10}$$

by assumption, and

$$M_{\mathbb{Z}} \xrightarrow{\cdot l} M_{\mathbb{Z}} \to H_{\mathbb{Z}/l} \tag{4.2.11}$$

induces a long exact sequence

$$\begin{tikzcd}
\left[\Sigma^{1,0} X, H_{\mathbb{Z}/l} \wedge F\right] \arrow[r] \arrow[dr, two heads] & {[X, M_{\mathbb{Z}} \wedge F]} \arrow[r, "\cdot l"] & {[X, M_{\mathbb{Z}} \wedge F]} \arrow[r] \arrow[dr, two heads] & \left[X, H_{\mathbb{Z}/l} \wedge F\right] \\
& \mathrm{Tor}^1([X, M_{\mathbb{Z}} \wedge F], \mathbb{Z}/l) \arrow[u, hook] & & {[X, M_{\mathbb{Z}} \wedge F] \otimes_{\mathbb{Z}} \mathbb{Z}/l} \arrow[u, hook]
\end{tikzcd}$$

which proves the statement. The rational acyclicity follows from [[NSØ09], Corollary 10.3], while the torsion case is a consequence of calculation using the motivic Steenrod algebra [[Hoy15], Theorem 5.17 & 6.19]. □

We need a fact from localization:

Lemma 4.2.4 *For any $M_{\mathbb{Z}}$-acyclic spectrum $E \in \mathcal{SH}(k)$ and* MGL*-localized connective spectrum $X \in \mathcal{SH}(k)$ we have $[E, X]_{\mathcal{SH}(k)} = 0$.*

Proof. This follows from the general theory of Bousfield localization [[Lur10], Lecture 20] and [[Man18], §5.1, §5.2], as we identify $M_{\mathbb{Z}}$- and MGL-localization both with η-completion defined later in Definition 5.1.4. □

Proof of Theorem 4.2.1. We show $F := \mathrm{fib}(\Lambda[1/e] \to M_{\mathbb{Z}}[1/e]) = 0$. By Lemma 4.2.3 F is $M_{\mathbb{Z}}$-acyclic and as Λ is a connective MGL-module (since $\mathcal{SH}(k)_{\geq 0}$ is closed under colimit and MGL is connective), by Lemma 4.2.4 we have

$$[F, \Lambda[1/e]] = 0. \tag{4.2.12}$$

Similarly as $M_{\mathbb{Z}}$ is a weak MGL-module by the orientation $v : \mathrm{MGL} \to M_{\mathbb{Z}}$ and $M_{\mathbb{Z}}$ is connective by [[Hoy15], Lemma 7.3], we know

$$[F, \Sigma^{-1,0} M_{\mathbb{Z}}[1/e]] = 0. \tag{4.2.13}$$

Now use the fiber sequence

$$\Sigma^{-1,0} M_{\mathbb{Z}}[1/e] \to F \to \Lambda[1/e] \to M_{\mathbb{Z}}[1/e] \tag{4.2.14}$$

and by (4.2.12) we have a section $F \to \Sigma^{-1,0} M_{\mathbb{Z}}[1/e]$ which is 0 by (4.2.13), therefore $F = 0$.

□

We also prove some consequences of this theorem.

Corollary 4.2.5 *[[Voe02], Conjecture 5] Let S be an essentially smooth scheme over a base field k. The slices of* MGL^* *in $\mathcal{SH}(S)$ are given by*

$$\mathrm{gr}^t \mathrm{MGL}^* \simeq \Sigma^{2t,t} HL_t[1/e] \tag{4.2.15}$$

where L_t is the t-th graded piece of L viewed as an Adams graded MU_**-module. In particular, for k a field of characteristic 0, the zeroth slice of* MGL *is $M_{\mathbb{Z}}$.*

Proof. For any essentially smooth morphism $f : S \to \mathrm{Spec}\ k$ of schemes we have $\mathrm{gr}^t f^* \simeq f^* \mathrm{gr}^t$ by essential smooth base change [[Hoy15], Lemma A.7]. Therefore, we may assume k is perfect. In this case, the statement follows from Theorem 4.2.1 and [[Spi10], Corollary 4.9].

□

In fact, this is true for any Landweber exact theory. Recall a graded $L[1/e]$-module M_* is called Landweber exact, if for every p prime, a regular sequence $v_0^{(p)}, v_1^{(p)}, v_2^{(p)}, \ldots$ where $v_n^{(p)}$ has degree $2(p^n - 1)$ in $L[1/e]$, is again regular in M_*. By [[NSØ09], Theorem 8.7], there is a spectrum $E \in \mathcal{SH}(k)$ representing the functor

$$\begin{aligned} \mathcal{SH}(k) &\to \mathrm{GrAb} \\ X &\mapsto \mathrm{MGL}_{**}(X) \otimes_{L[1/e]} M_*. \end{aligned} \tag{4.2.16}$$

Theorem 4.2.6 *[[Spi12], Theorem 6.1] Let M_* be a Landweber exact $L[1/e]$-module and E the associated spectrum in $\mathcal{SH}(k)$. There is a unique equivalence of $M_{\mathbb{Z}}$-modules:*

$$\mathrm{gr}^t(E^*) \simeq \Sigma^{2t,t} H_{M_t} \tag{4.2.17}$$

making the diagram

$$\begin{tikzcd} \pi_{0,0}\mathrm{MGL} \otimes M_t \arrow[r, "v"] \arrow[d] & \pi_{0,0} H_{M_t} \arrow[d, "\cong"'] \\ \pi_{2t,t} E \arrow[r] & \pi_{2t,t}(\mathrm{gr}^t(E^*)) \end{tikzcd}$$

commute. Here H_{M_t} is the motivic Eilenberg-MacLane spectrum associated to the group M_t.

As a motivic spectrum constructed in Definition 3.4.14, $\mathrm{BP}^{(\ell)}_{\mathrm{mot}}$ is the direct summand of $\mathrm{MGL}_{(\ell)}$ killing all ℓ-typical formal group laws. A direct consequence of Theorem 4.2.1 shows that it takes the form $\mathrm{MGL}_{(\ell)}/x$ where $x \in L$ is a regular sequence generating the vanishing ideal of all ℓ-typical formal group laws [[Hoy15], Example 8.13]. We have the following corollary:

Corollary 4.2.7 $\mathrm{BP}^{(\ell)}_{\mathrm{mot}}$ *is a Landweber exact theory and we have*

$$\mathrm{gr}^0 \mathrm{BP}^{(\ell)}_{\mathrm{mot}} \simeq H\Big(L_{(\ell)}[1/e]\Big)_0 \simeq M_{\mathbb{Z}}[1/e] \otimes \mathbb{Z}_{(\ell)}. \tag{4.2.18}$$

Another beautiful consequence is regarding the general slice filtration of a Landweber exact spectrum.

Theorem 4.2.8 *[[Hea19], Proposition 4.11] For any Landweber exact spectrum $E \in \mathcal{SH}(k)$, the slice filtration and generalized slice filtration of E agree.*

Proof. We prove the theorem for MGL, the rest is a comparison of homotopy groups via base change. We want to show $\mathrm{MGL}^n \in \mathcal{SH}^{\mathrm{veff}}(k)(n)$. By Theorem 4.2.1 and [[Spi10], Theorem 4.7], we can express MGL^n as the colimit of a diagram of $\mathrm{MGL}[1/e]$-modules like $\Sigma^{2m,m}\mathrm{MGL}[1/e]$ for $m \geq n$. By Theorem 3.3.2 and construction, MGL is a very effective spectrum, hence $\Sigma^{2m,m}\mathrm{MGL}[1/e] \in \mathcal{SH}^{\mathrm{veff}}(k)(m)$ by definition. Finally since $\mathcal{SH}^{\mathrm{veff}}(k)(m) \subset \mathcal{SH}^{\mathrm{veff}}(k)(n)$ is closed under colimits we conclude. □

4.3 Realization functor

Realization functors are colimit preserving exact functors that send motivic spectra into a simpler stable ∞-categories, e.g. $\mathcal{SH}$. We will introduce three kinds of realization functors: the complex Betti realization, the real Betti realization and the étale realization functor. We begin with the first two cases.

Let k be a field of characteristic zero equipped with an embedding $k \hookrightarrow \mathbb{C}$. To construct the complex realization functor, we lift the functor $(-)(\mathbb{C}) : \mathcal{S}\mathrm{m}_k \to \mathrm{An}$, which associates a smooth scheme S with the homotopy type of the space of its complex points $S(\mathbb{C})$ under the analytic topology, to a colimit preserving functor $\mathrm{Re}_{\mathbb{C}} : \mathcal{P}(\mathcal{S}\mathrm{m}_k) \to \mathrm{An}$ by the universal property of presheaves Theorem 6.5.3.

Now we put an eye on the compactability of $(-)(\mathbb{C})$ with Nisnevich excision. Since étale morphism induces a locally split map over analytic topology, we have precisely the following proposition:

Proposition 4.3.1 *[[Voe10], Lemma 3.38] Let $\left\{U \xrightarrow{i} X, V \xrightarrow{p} X\right\}$ be a Nisnevich square as defined in Definition 3.1.2, then $X(\mathbb{C}) \simeq U(\mathbb{C}) \coprod_{(U\times V)(\mathbb{C})} V(\mathbb{C})$ is a homotopy equivalence.*

Since $(-)(\mathbb{C})$ preserves products, we have

$$(S \times \mathbb{A}^1_k)(\mathbb{C}) \simeq S(\mathbb{C}) \times \mathbb{A}^1_k(\mathbb{C}) \simeq S(\mathbb{C}) \times \mathbb{C} \simeq S(\mathbb{C}) \tag{4.3.1}$$

and together with the previous proposition, we conclude that $\mathrm{Re}_{\mathbb{C}} : \mathcal{P}(\mathcal{S}\mathrm{m}_k) \to \mathrm{An}$ factors through $\mathcal{H}(k)$ of motivic spaces. The same construction works for pointed spaces $\mathcal{H}(k)_*$.

The final step is to post-compose the stabilization functor and check whether $\Sigma^\infty \circ \mathrm{Re}_{\mathbb{C}} : \mathcal{H}(k)_* \to \mathcal{SH}$ is symmetric monoidal and inverts $\mathbb{P}^1$. To see why this suffices, notice that by construction $\mathcal{SH}(k)$ is the universal symmetric monoidal ∞-category that is $\mathbb{P}^1$-stable as in Proposition 3.2.11. Therefore, such a functor must factor through $\mathcal{SH}(k)$.

Proposition 4.3.2 *The functor $\Sigma^\infty \circ \mathrm{Re}_{\mathbb{C}} : \mathcal{H}(k)_* \to \mathcal{SH}$ is symmetric monoidal and $\Sigma^\infty(\mathrm{Re}_{\mathbb{C}}(\mathbb{P}^1))$ is a $\otimes$-invertible spectrum.*

Proof. By Proposition 6.4.6, we need to verify that $\mathrm{Re}_{\mathbb{C}}$ preserves finite products. As a right adjoint, localization preserves products, and since every presheaf is a colimit of representable objects, we see this is true as $\mathrm{Re}_{\mathbb{C}}$ is colimit preserving and $(-)(\mathbb{C})$ preserves products.

We compute

$$\mathrm{Re}_{\mathbb{C}}(\mathbb{P}^1) \simeq \mathrm{Re}_{\mathbb{C}}(\mathbb{S}^1) \wedge \mathrm{Re}_{\mathbb{C}}(\mathbb{G}_m) \simeq \mathbb{S}^1 \wedge \mathbb{C}^\times \simeq \mathbb{S}^2 \tag{4.3.2}$$

where we view $\mathbb{S}^1$ as $\Sigma(* \coprod *)$ and use that $\mathrm{Re}_{\mathbb{C}}$ is colimit preserving. Since $\mathbb{S}^2$ is indeed invertible in $\mathcal{SH}$ we conclude. $\square$

Corollary 4.3.3 *The functor $\Sigma^\infty \circ \mathrm{Re}_{\mathbb{C}} : \mathcal{H}(k)_* \to \mathcal{SH}$ factors through $\mathcal{SH}(k)$ and induces a functor $\mathrm{Re}_{\mathbb{C}} : \mathcal{SH}(k) \to \mathcal{SH}$, which is well-defined, symmetric monoidal, preserves colimits and finite products. We call this functor complex Betti realization.*

Remark 4.3.4 Notice $\mathrm{Re}_{\mathbb{C}}$ preserves the $\mathcal{E}_n$-ring structure on $\mathcal{SH}(k)$ and $\mathcal{SH}$ for $1 \leq n \leq \infty$, this is a direct consequence of the fact that $\mathrm{Re}_{\mathbb{C}}$ is lax symmetric monoidal.

We want to understand the $\mathbb{Z}/2$-action on $\mathbb{C}$-points of a scheme. We claim it suffices to consider only $\mathbb{R}$-schemes. Indeed, if k embeds into $\mathbb{R}$, then the embedding $\alpha : k \hookrightarrow \mathbb{R}$ induces a base change functor $\mathcal{S}\mathrm{m}_k \to \mathcal{S}\mathrm{m}_{\mathbb{R}}$, which in turns gives us a bijection $\mathcal{S}\mathrm{m}_k(\mathrm{Spec}(\mathbb{R}), S) \simeq \mathcal{S}\mathrm{m}_{\mathbb{R}}(\mathrm{Spec}(\mathbb{R}), S \times_k \mathrm{Spec}(\mathbb{R}))$ for each $S \in \mathcal{S}\mathrm{m}_k$.

The set $\mathbb{C}$-points of a $\mathbb{R}$-scheme naturally carries a $\mathbb{Z}/2$-action by complex conjugation, hence similarly, the functor $(-)(\mathbb{C})$ pre-composing with base change induces a functor $\mathrm{Re}_{\mathbb{R}} : \mathcal{H}(k) \to \mathcal{P}\left(\mathcal{O}_{\mathbb{Z}/2}\right)$, where $\mathcal{O}_{\mathbb{Z}/2}$ is the orbit category of $\mathbb{Z}/2$.

Corollary 4.3.5 *The functor $\Sigma^\infty \circ \mathrm{Re}_{\mathbb{R}} : \mathcal{H}(k)_* \to \mathcal{SH}_{C_2}$, where $\mathcal{SH}_{C_2}$ is the category of genuine C_2 -spectra, factors through $\mathcal{SH}(k)$ and induces a functor $\mathrm{Re}_{\mathbb{R}} : \mathcal{SH}(k) \to \mathcal{SH}_{C_2}$, which is well-defined, symmetric monoidal, preserves colimits and finite products. We call this functor real Betti realization.*

The following proposition explains the motivation of defining very effective spectra and the t -exactness of the realization functor.

Proposition 4.3.6 *The restriction of $\mathrm{Re}_{\mathbb{C}} : \mathcal{SH}^{\mathrm{veff}}(k)(m) \to \mathcal{SH}_{\geq 2m}$ is well-defined, where $\mathcal{SH}_{\geq 2m}$ is the subcategory of $2m$-connected spectra.*

Proof. For any $X_+ \in \mathcal{H}(k)_*$, we have $\mathrm{Re}_{\mathbb{C}}(\Sigma^\infty_{\mathbb{P}^1} X_+) \simeq \Sigma^\infty \mathrm{Re}_{\mathbb{C}}(X_+)$ by definition, clearly this is connected. Now for each $(\mathbb{P}^1)^{\wedge m} \wedge \Sigma^\infty_{\mathbb{P}^1} X_+$,

$$\mathrm{Re}_{\mathbb{C}}\left((\mathbb{P}^1)^{\wedge m} \wedge \Sigma^\infty_{\mathbb{P}^1} X_+\right) \simeq \mathbb{S}^{2m} \wedge \Sigma^\infty \mathrm{Re}_{\mathbb{C}}(X_+) \tag{4.3.3}$$

is $2m$-connected since $\mathrm{Re}_{\mathbb{C}}$ is symmetric monoidal. The statement now follows from the fact that connected spectra are closed under colimits. □

Remark 4.3.7 The proof works similarly for real realization functor which takes value in m-connected genuine C_2-spectra.

Until the end of this section we fix an algebraically closed field k of characteristic p.

The situation in étale case is a bit tricky, as the étale homotopy type taken in p-local coefficients is not $\mathbb{A}^1$-invariant in general, and the étale homotopy type functor due to Artin-Mazur-Friedlander is also not product-preserving.

We therefore work away from the characteristic and give an ∞-categorical definition of the étale realization following [Hoy18].

Let $\mathrm{An}^\wedge$ be the category of profinite spaces, it is naturally identified with pro-objects of presheaves of Fin finite sets via the limit functor. For any shape $S \in \mathrm{Pro}(\mathrm{An})$, we can associate a profinite completion $S^\wedge \in \mathrm{An}^\wedge$ to it. Our target of étale realization functor will be the $\mathbb{S}^2$-stabilization of $\mathrm{An}^\wedge_*$, viewing $\mathbb{S}^2$ as a constant sheaf, to which we denote $\mathcal{SH}^{\wedge,\mathbb{S}^2}$.

Now for any scheme X, let $\mathrm{Ét}_X$ be the étale site over X and $\mathcal{X}_{\text{ét}} := \mathrm{Shv}_{\text{ét}}\left(\mathrm{Ét}_X\right)$ be the ∞-topos of étale sheaves on X, let $\mathcal{X}^\wedge_{\text{ét}}$ be its hypercompletion. As the geometric morphism $\mathcal{X}_{\text{ét}} \to \mathcal{X}_{\mathrm{Zar}}$

to Zariski sheaves induces an equivalence on (-1)-truncated objects, we see if X is locally Noetherian, then $\mathcal{X}_{\text{ét}}$ is locally connected and we may apply the following theorem:

Proposition 4.3.8 *[[Hoy18], Corollary 5.6] Let X be a scheme such that $\mathcal{X}_{\text{ét}}$ is locally connected, then the étale topological type of X corepresents the fundamental pro-∞-groupoid $\Pi_\infty \mathcal{X}_{\text{ét}}^\wedge$ as defined in Definition 6.7.3.*

As the fundamental pro-∞-groupoid is a shape, we have a functor $\mathrm{Re}_{\text{ét}} : \mathrm{Shv}_{\text{ét}}(\mathcal{S}\mathrm{m}_S) \to \mathrm{Pro}(\mathrm{An})$ and by composing with completion and stabilization, a functor $\mathrm{Re}_{\text{ét}} : \mathrm{Shv}_{\text{ét}}(\mathcal{S}\mathrm{m}_S) \to \mathcal{SH}^{\wedge,\mathbb{S}^2}$.
Since étale topology is finer than Nisnevich topology, every étale sheaf is a Nisnevich sheaf, and we have the realization functor $\mathrm{Re}_{\text{ét}} : \mathrm{Shv}_{\mathrm{Nis}}(\mathcal{S}\mathrm{m}_S) \to \mathcal{SH}^{\wedge,\mathbb{S}^2}$.
To get an $\mathbb{A}^1$-invariant functor, we have to pick a prime $\ell \neq \mathrm{char}(k)$ of the base field and localize $\mathrm{Pro}(\mathrm{An})$ with respect to all pro-morphisms that induce isomorphisms on objectwise continuous cohomology with $\mathbb{Z}/\ell$-coefficients. This is a reasonable localization as suggested by [[Isa04], Theorem 2.5]. We denote it as $\mathrm{Pro}(\mathrm{An})_\ell$.
Luckily, away from characteristic the étale cohomology is $\mathbb{A}^1$-invariant as [[Sta25], Tag 03SB], and by [[Fri82], Proposition 5.9] the realization functor extends to $\mathcal{H}(S) \to \mathrm{Pro}(\mathrm{An})_\ell \to \mathcal{SH}^{\wedge,\mathbb{S}^2}$.

Proposition 4.3.9 *We have* $\mathrm{Re}_{\text{ét}}(\mathbb{P}^1_k) \cong \Sigma^\infty \mathbb{S}^2$.

Proof. This follows from the facts $\pi_1^{\text{ét}}(\mathbb{P}^1_k) = 0$ and (notice $\mathrm{Br}(k) = 0$):

$$H^*_{\text{ét}}(\mathbb{P}^1_k; \mathbb{Z}/\ell) \cong \begin{cases} \mathbb{Z}/\ell & \text{if } * = 0,\ 2 \\ 0 & \text{else} \end{cases} \tag{4.3.4}$$

with a chosen isomorphism $\mu_\ell \to \mathbb{Z}/\ell$. $\square$

Corollary 4.3.10 *The functor $\mathrm{Re}_{\text{ét}} : \mathcal{SH}(k) \to \mathcal{SH}^{\wedge,\mathbb{S}^2}$ is a well-defined exact functor. We call this functor étale realization.*

Remark 4.3.11 Many classical topological spectra can be completed to pro-$\mathbb{S}^2$-spectra via completion, for example, $\mathrm{MU} \mapsto \widehat{\mathrm{MU}}$. These profinite spectra represents cohomology theories: Let $X \in \mathcal{S}\mathrm{m}_S$ and $E \in \mathcal{SH}^{\wedge,\mathbb{S}^2}$, then the n-th E-étale cohomology is

$$E^n_{\text{ét}}(X) := [\mathrm{Re}_{\text{ét}}(X), \Sigma^{-n} E]_{\mathcal{SH}^{\wedge,\mathbb{S}^2}}. \tag{4.3.5}$$

In particular, the étale cohomology theory representing $\widehat{\mathrm{MU}}$ is called étale cobordism as defined in [[Qui07], §4.2.3].

4.4 Realizations and slices

We collect some realizations of motivic spectra and their slices.

Theorem 4.4.1 *Let* MU *denote the complex cobordism spectrum in* $\mathcal{SH}$. *Then there is an isomorphism of* $\mathcal{E}_\infty$ *rings:* $\mathrm{Re}_\mathbb{C}\mathrm{MGL} \simeq \mathrm{MU}$, *where* MU *is equipped with the* $\mathcal{E}_\infty$ *structure as a Thom spectrum.*

Proof. In order to show that the realization of the motivic Thom spectrum is equivalent to the topological Thom spectrum as an $\mathcal{E}_\infty$-ring, it suffices to show that they induce the same symmetric monoidal structure on MU.

We observe the construction in Theorem 3.5.3 can be extended to any colimit preserving functor $F : \mathrm{Span} \to \mathrm{Cat}_\infty^{\mathrm{sift}}$ as pointed out in [[Ban25], §2.4.2]. Fix a cardinal κ. We set F to be the functor sending $X \in \mathcal{S}\mathrm{m}_k$ to $\mathcal{SH}\left(\mathrm{An}_{/\,\mathrm{Re}_\mathbb{C}(X)}\right)^\kappa$, i.e. the spectra over $\mathrm{Re}_\mathbb{C}(X)$ that are closed under κ-small colimits. By a slight modification of [[Ban25], Proposition 2.36] (replacing $\mathbb{R}$ with $\mathbb{C}$), this gives us a symmetric monoidal functor

$$M_\mathbb{C} : \mathcal{P}_\Sigma(\mathcal{S}\mathrm{m}_k)_{/\,F^\simeq} \to F(\mathbb{C}) \simeq \mathcal{SH} \tag{4.4.1}$$

and a natural transformation $\alpha : \mathcal{SH} \to F$ such that

$$\begin{tikzcd}
\mathcal{P}_\Sigma(\mathcal{S}\mathrm{m}_k)_{/\mathcal{SH}^\simeq} \arrow[r, "\alpha_\sharp"] \arrow[d, "M_k"'] & \mathcal{P}_\Sigma(\mathcal{S}\mathrm{m}_k)_{/\,F^\simeq} \arrow[d, "M_\mathbb{C}"'] \\
\mathcal{SH}(k) \arrow[r, "\alpha_\mathbb{C}"] & \mathcal{SH}
\end{tikzcd}$$

commutes. For any $X \in \mathcal{S}\mathrm{m}_k$, α_X sends the structure map $X \to \mathrm{Spec}\ k$ to the complex Betti realization $\mathrm{Re}_\mathbb{C}(\Sigma^\infty_{\mathbb{P}^1} X_+) \to \mathbb{S}$ and this map is symmetric monoidal. Since complex Betti realization factors through $\mathbb{A}^1$-invariant Zariski sheaf and commutes with the $\mathbb{A}^1$-localization as by [[Ayo10], Theoreme 4.9], the following diagram

$$\begin{tikzcd}
\mathcal{P}_\Sigma(\mathcal{S}\mathrm{m}_k)_{/\mathcal{SH}^\simeq} \arrow[r, "\alpha_\sharp"] \arrow[d, "M_k"'] & \mathcal{P}_\Sigma(\mathcal{S}\mathrm{m}_k)_{/\,F^\simeq} \arrow[r] \arrow[d, "M_\mathbb{C}"'] \arrow[rr, bend left, "\mathrm{Re}_\mathbb{C}"] & L_{\mathbb{A}^1}\mathrm{Shv}_{\mathrm{Zar}}(\mathcal{S}\mathrm{m}_k)_{/\,F^\simeq} \arrow[r] & \mathrm{An}_{/\,\mathrm{Sp}^\simeq} \arrow[dll, "M'"] \\
\mathcal{SH}(k) \arrow[r, "\alpha_\mathbb{C} \simeq \mathrm{Re}_\mathbb{C}"] & \mathcal{SH} & &
\end{tikzcd}$$

is a commutative diagram of symmetric monoidal functors, where Sp is the stabilization functor (in order to distinguish with our notation of $\mathbb{P}^1$-stabilization $\mathcal{SH}$) and M' is our candidate of topological Thom spectrum functor. But the argument in [[Ban25], Lemma 2.45] applies and we can identity M' with M the topological Thom spectrum functor.

From $\mathrm{Re}_\mathbb{C} \circ M_k \simeq M \circ \mathrm{Re}_\mathbb{C} \circ \alpha_\sharp$ this reduces to job to check whether motivic j-homomorphism composing with α is the topological Thom spectrum functor. Recall MU is identified with $M\left(\mathrm{BU} \xrightarrow{j} \mathcal{SH}^\simeq\right)$ and $\mathrm{MGL} = M_k\left(K^\circ \xrightarrow{j} \mathcal{SH}^\simeq\right)$, so we reduce to check $\mathrm{Re}_\mathbb{C}(K^\circ) \simeq \mathrm{BU}$. To see this, one compute

$$\begin{aligned}\mathrm{Re}_{\mathbb{C}}(K^{\circ}) &\simeq \mathrm{Re}_{\mathbb{C}}(\mathrm{BGL}) = \mathrm{Re}_{\mathbb{C}}(\mathrm{colim}_{\Delta^{\mathrm{op}}}(\mathrm{GL} \to \mathrm{GL} \times \mathrm{GL} \to \dots)) \\ &\underset{1}{\simeq} \mathrm{B}\ (\mathrm{Re}_{\mathbb{C}}(\mathrm{GL})) = \mathrm{B}\left(\bigcup_{n} \mathrm{GL}_n(\mathbb{C})\right) \underset{2}{\simeq} \mathrm{B}\big(\bigcup U_n(\mathbb{C})\big) = \mathrm{BU}\end{aligned} \tag{4.4.2}$$

where 1 is true since $\mathrm{Re}_{\mathbb{C}}$ preserves colimit and 2 comes from the deformation retract $U_n(\mathbb{C}) \hookrightarrow \mathrm{GL}_n(\mathbb{C})$. □

Corollary 4.4.2 *Let k be a field of characteristic 0 equipped with an embedding $k \hookrightarrow \mathbb{C}$. Let MU^* be the Postnikov filtration on MU and MGL^* the slice filtration on $\mathrm{MGL} \in \mathcal{SH}(k)$, then $\mathrm{Re}_{\mathbb{C}}(\mathrm{MGL}^n)$ is $(2n-1)$-connected, and we have an isomorphism in $\mathcal{SH}$:*

$$\tau_{\geq 2n}\mathrm{Re}_{\mathbb{C}}(\mathrm{MGL}^n) \cong \mathrm{MU}^{2n} \tag{4.4.3}$$

Proof. By Theorem 4.2.8 and Proposition 4.3.6, it is enough to verify that MGL is (-1)-connected, which follows from Theorem 3.3.2 and construction.

For the second assertion, we truncate the slice filtration to $2n$-level and consider the two strong convergent spectral sequences associated to

$$\begin{gathered}\dots \to \mathrm{MGL}^{m+1} \to \mathrm{MGL}^m \to \dots \to \mathrm{MGL} \\ \dots \to \mathrm{MGL}^{m+1} \to \mathrm{MGL}^m \to \dots \to \mathrm{MGL}^{2n}.\end{gathered} \tag{4.4.4}$$

Together with Theorem 4.4.1 this shows that

$$\pi_m(\mathrm{MU}) \cong \pi_m(\mathrm{Re}_{\mathbb{C}}(\mathrm{MGL})) \cong \pi_m(\mathrm{Re}_{\mathbb{C}}(\mathrm{MGL}^n)) \tag{4.4.5}$$

for $m \geq 2n$. Therefore $\mathrm{Re}_{\mathbb{C}}(\mathrm{MGL}^n)$ is a $(2n-1)$-connected cover of MU, hence the statement is true. □

Since $\mathrm{Re}_{\mathbb{C}}$ preserves colimit and finite products, and $\mathrm{BP}^{(\ell)}_{\mathrm{mot}}$ is a direct summand of $\mathrm{MGL}_{(\ell)}$, by Theorem 4.4.1 we have:

Corollary 4.4.3 *For ℓ a prime we have $\mathrm{Re}_{\mathbb{C}}\mathrm{BP}^{(\ell)}_{\mathrm{mot}} \simeq \mathrm{BP}^{(\ell)}$, where $\mathrm{BP}^{(\ell)} \in \mathcal{SH}$ is the classical Brown-Peterson spectrum.*

Theorem 4.4.4 *[[Ban25], Theorem 2.48] Let k be a field of characteristic 0 equipped with an embedding $k \hookrightarrow \mathbb{R}$. Let MO denote the oriented cobordism spectrum in $\mathcal{SH}_{C_2}$. Then there is an isomorphism of $\mathcal{E}_\infty$ rings:* $\mathrm{Re}_{\mathbb{R}}\mathrm{MGL} \simeq \mathrm{MO}$.

Proof. Replace the complex realization functor in the proof of Theorem 4.4.1 with real realization functor, we deduce that the real realization of motivic Thom spectrum functor agrees with the topological Thom spectrum functor as explained in [[Ban25], Theorem 2.47]. Hence it suffices to check $\mathrm{Re}_{\mathbb{R}}(\mathrm{BGL}) \simeq \mathrm{BO}$. But as $O_n(\mathbb{R}) \hookrightarrow \mathrm{GL}_n(\mathbb{R})$ is a deformation retract, this is obvious.

□

In general the interaction between slices and Betti realization functor could be very complicated, the next two theorems are about the realization of slices of the motivic sphere spectrum.

Theorem 4.4.5 *[[Lev14], Lemma 6.4] Let k be a field of characteristic 0 equipped with an embedding $k \hookrightarrow \mathbb{C}$. Suppose k has finite cohomological dimension, then for all n and q, the complex Betti realization induces an isomorphism*

$$\mathrm{Re}_{\mathbb{C},*} : \pi_{n,0}(\mathrm{gr}^q \mathbb{S}_k^*)(k) \xrightarrow{\cong} \pi_n(\mathrm{Re}_{\mathbb{C}}(\mathrm{gr}^q \mathbb{S}_k^*)) \tag{4.4.6}$$

Corollary 4.4.6 *Let k be an algebraically closed field of characteristic 0. Fix an embedding $\sigma : k \hookrightarrow \mathbb{C}$, then $\pi_{n,0}(\mathbb{S}_k)(k) \cong \pi_n(\mathbb{S})$ for all n.*

Proof. This is a standard spectral sequence argument. Notice $\mathrm{Re}_{\mathbb{C}}$ is symmetric monoidal and $\mathrm{Re}_{\mathbb{C}}(\mathbb{S}_k) \simeq \mathbb{S}$. By Theorem 5.1.8 there is a convergent spectral sequence

$$\pi_{n,0}(\mathrm{gr}^q \mathbb{S}_k^*)(k) \Rightarrow \pi_{n,0}(\mathbb{S}_k)(k) \tag{4.4.7}$$

and the slice induces a spectral sequence

$$\pi_n(\mathrm{Re}_{\mathbb{C}}(\mathrm{gr}^q \mathbb{S}_k^*)) \Rightarrow \pi_n(\mathrm{Re}_{\mathbb{C}}(\mathbb{S}_k)) \cong \pi_n(\mathbb{S}). \tag{4.4.8}$$

Invoke Theorem 4.4.5 and we have the desired result. □

The next example shows that the complex and real Betti realization can be very different.

Theorem 4.4.7 *Let k be a field of characteristic 0 equipped with an embedding $k \hookrightarrow \mathbb{R}$, then:*

1. *$\mathrm{Re}_{\mathbb{C}}\mathrm{KGL} \cong \mathrm{KU}$, the spectrum representing complex topological K-theory.*
2. *$\mathrm{Re}_{\mathbb{R}}\mathrm{KGL} = 0$.*

Proof. By Proposition 4.3.1 the realization functors satisfy Nisnevich descents, hence Definition 3.4.6 tells us $\mathrm{Re}_{\mathbb{C}}(L_{\mathrm{mot}}(\mathbb{Z} \times \mathrm{BGL})) \simeq \mathrm{Re}_{\mathbb{C}}(\mathbb{Z} \times \mathrm{BGL}) \simeq \mathbb{Z} \times \mathrm{BU}$ by the same computation in Theorem 4.4.1.

Now as $\mathrm{Re}_{\mathbb{C}}(\mathbb{P}^1) \simeq \mathbb{S}^2$, the structure map after realization is $\mathbb{S}^2 \times \mathrm{Re}_{\mathbb{C}}(E_n) \to \mathrm{Re}_{\mathbb{C}}(E_{n+1})$ for a motivic spectrum $E = (E_0, E_1, \ldots)$. Since $\mathrm{Re}_{\mathbb{C}}$ commutes with colimits, in particular, it commutes with the spectrification functor $\mathcal{SH}^{\mathrm{pre}} \to \mathcal{SH}$, we see that the complex realization of KGL should be the $\mathbb{S}^2$-spectrification of

$$(\mathbb{Z} \times \mathrm{BU}, \mathbb{Z} \times \mathrm{BU}, \ldots) \tag{4.4.9}$$

which is also the $\mathbb{S}^2$-spectrification of $\mathrm{KU} = (\mathbb{Z} \times \mathrm{BU}, \mathrm{U}, \mathbb{Z} \times \mathrm{BU}, U, \ldots)$ by complex Bott periodicity.

Similarly, $\mathrm{Re}_{\mathbb{R}}(\mathrm{KGL})$ should be the $\mathbb{S}^1$-spectrification of

$$(\mathrm{Re}_{\mathbb{R}}(\mathbb{Z} \times \mathrm{BGL}), \mathrm{Re}_{\mathbb{R}}(\mathbb{Z} \times \mathrm{BGL}), \ldots) \tag{4.4.10}$$

of which we have calculated in Theorem 4.4.4:

$$\mathrm{Re}_{\mathbb{R}}(\mathbb{Z} \times \mathrm{BGL}) \simeq \mathbb{Z} \times \mathrm{BO}. \tag{4.4.11}$$

However, the real Bott periodicity tells us that $\pi_n(\mathbb{Z} \times \mathrm{BO})$ is the 8-periodic sequence $\mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}, 0, 0, 0$, so the spectrification must be 0. □

5 Application: Spectral sequences

5.1 Motivic Atiyah-Hirzebruch spectral sequence

As we have already seen in §2.3, each filtration induces a slice spectral sequence. We thus give the following definition.

Definition 5.1.1 Let $E \in \mathcal{SH}(S)$ and E^* the slice filtration on E. Then Construction 2.3.10 yields a spectral sequence related to E^*, we call this the motivic slice spectral sequence associated to E^*.

Remark 5.1.2 We are primarily interested in the global section of this slice spectral sequence. This is also what Levine is referring to as motivic Atiyah-Hirzebruch spectral sequence:

$$E_1^{s,t}(AH)(E) := \pi_{s+t,0}(\mathrm{gr}^s E^*)(\mathrm{Spec}\ k) \Rightarrow \pi_{s+t,0}(E)(\mathrm{Spec}\ k) \tag{5.1.1}$$

We have also the ℓ-local and ℓ-complete version of this spectral sequence whenever ℓ is a prime different from the characteristic of k.

Definition 5.1.3 Let $E \in \mathcal{SH}(S)$ and ℓ a prime number. We have the ℓ-localization $E_{(\ell)}$ and ℓ-completion $E_\ell^\wedge$ with slice filtrations $E_{(\ell)}^*$ and $(E_\ell^\wedge)^*$ respectively. We call the from $E_{(\ell)}^*$ and $(E_\ell^\wedge)^*$ induced spectral sequences ℓ-local and ℓ-complete motivic slice spectral sequences associated to E.

The convergence of the motivic slice spectral sequence was conjectured in [Voe02]. In order to formulate the best convergence result established so far, we introduce some concepts regarding the completion of motivic spectra.

Let $E \in \mathcal{SH}(S)$ be a spectrum and let E^* be the associated slice filtration. We define an endofunctor

$$\mathrm{sc} : \mathcal{SH}(S) \to \mathcal{SH}(S), E \mapsto \mathrm{cofib}\Big(\lim_n E^n \to E\Big) \tag{5.1.2}$$

and call this functor the slice completion of E^*. Clearly the slice filtration is complete if and only if $\mathrm{sc}(E) \simeq 0$.

Definition 5.1.4

1. The algebraic Hopf map is the class $\eta \in \pi_{1,1}(\mathbb{S}_S)$ induced by the coordinate map
$$\mathbb{A}_S^2 - \{0\} \to \mathbb{P}_S^1, (x, y) \mapsto [x : y]. \tag{5.1.3}$$
2. The limit of the sequential tower
$$\ldots \to E \wedge \mathrm{cofib}(\eta^{n+1}) \to E \wedge \mathrm{cofib}(\eta^n) \to E \wedge \mathrm{cofib}(\eta^{n-1}) \to \ldots \tag{5.1.4}$$
is called the η-completion $E_\eta^\wedge$ of E.

We denote $\mathcal{SH}^{\text{cell}}(S)$ to be the smallest subcategory of $\mathcal{SH}(S)$ which contains $\mathbb{S}_S$ and is closed under small colimits. We call this the category of cellular motivic spectra over S. The slice completion is closely related to the η-completion of a cellular spectrum.

Proposition 5.1.5 *[[Man18]. §5.3] Let $E \in \mathcal{SH}(S)$ be a cellular spectrum of finite type. Let E_{MGL} and $E_{M_{\mathbb{Z}}}$ be the* MGL*- and $M_{\mathbb{Z}}$-localization of E respectively. Then*

$$\mathrm{sc}(E) \simeq E^{\wedge}_{\eta} \simeq E_{\mathrm{MGL}} \simeq E_{M_{\mathbb{Z}}}. \tag{5.1.5}$$

The next theorem establishes the convergence of the slice spectral sequence over certain fields.

Theorem 5.1.6 *[[BEØ24], Corollary 5.10] Let k be a field of exponential characteristic e and $t > 0$ coprime to e such that $\mathrm{vcd}_t(k) < \infty$. Suppose $E \in \mathcal{SH}(k)_{\geq c}$ for some $c \in \mathbb{Z}$.*

1. *The map $\left(E^{\wedge}_{t,\rho}\right)^* \to \mathrm{sc}\left(E^{\wedge}_{t,\rho}\right)$ is an isomorphism on $\pi_{*,*}$.*
2. *There is a conditionally convergent spectral sequence*

$$\pi_{p,n}\left(\mathrm{gr}^q\left(E^{\wedge}_{t,\rho}\right)^*\right) \Longrightarrow \pi_{p,n}\left(E^{\wedge}_{t,\rho}\right) \tag{5.1.6}$$

where ρ is the endofunctor of smashing with $\mathbb{G}_m$.

Remark 5.1.7 Suppose $\mathrm{cd}_2(k) < \infty$, then $\rho^m \simeq 0$ for some m, hence, then ρ-completion of E is easy to compute. In fact in this case, for $x \in X \in \mathcal{S}\mathrm{m}_k$ we have

$$\pi_{i,j}(E^M)_x = 0 \tag{5.1.7}$$

for some $M \gg 0$ and the spectral sequence degenerates.

On taking the global section, the motivic Atiyah-Hirzebruch spectral sequence always converges for k a perfect field:

Theorem 5.1.8 *[[Lev13], Theorem 7.3] Let k a perfect field of exponential characteristic e with finite cohomological dimension. Let ℓ be coprime to e. Then we have a strongly convergent spectral sequence*

$$\pi_{p,n}(\mathrm{gr}^q(E^{\wedge}_{\ell})^*)(k) \Rightarrow \pi_{p,n}(E^{\wedge}_{\ell})(k). \tag{5.1.8}$$

However, as we will see in the next section, the convergence of the Atiyah-Hirzebruch spectral sequence associated to $\mathbb{S}_k$ (and MGL) can also be shown by relating it to the convergence of classical Adams-Novikov spectral sequences.

5.2 Comparison theorem

Let k be an algebraically closed field of characteristic zero. Fix an embedding $\sigma : k \hookrightarrow \mathbb{C}$ and let $\mathrm{Re}_{\mathbb{C}} : \mathcal{SH}(k) \to \mathcal{SH}$ be the associated complex Betti realization functor.
The goal of this section is to reprove the main result of [Lev15], which, adapted to our notations, has the following formulation:

Theorem 5.2.1 *Consider the Adams-Novikov spectral sequence*

$$E_2^{s,t}(AN) = \mathrm{Ext}^{s,t}_{\mathrm{MU}_*(\mathrm{MU})}(\mathrm{MU}_*, \mathrm{MU}_*) \Longrightarrow \pi_{t-s}\mathbb{S} \tag{5.2.1}$$

and the motivic Atiyah-Hirzebruch spectral sequence

$$E_1^{p,q}(AH) = \pi_{-p-q,0}(\mathrm{gr}^{-q}\mathbb{S}_k^\star)(k) \Longrightarrow \pi_{-p-q,0}(\mathbb{S}_k)(k). \tag{5.2.2}$$

Then there is an isomorphism

$$\gamma_1^{p,q} : E_1^{p,q}(AH) \cong E_2^{3p+q,2p}(AN) \tag{5.2.3}$$

which induces a sequence of isomorphisms of complexes for $r \geq 1$

$$\oplus_{p,q}\, \gamma_r^{p,q} : \left(\oplus_{p,q}\, E_r^{p,q}(AH), d_r\right) \to \left(\oplus_{p,q}\, E_{2r+1}^{3p+q,2p}(AN), d_{2r+1}\right). \tag{5.2.4}$$

To establish the isomorphism at E_1-page, one proves the following key lemma.

Lemma 5.2.2 *The Betti realization functor gives an isomorphism*

$$\mathrm{Re}(\mathrm{gr}^{[a,b)}\mathbb{S}_k^\star) \cong \mathrm{gr}^{[2a,2b)}\mathrm{Dec}^\bullet(\mathrm{MU}^{\wedge *+1}) \tag{5.2.5}$$

where Dec *is the décalage functor introduced in* §2.3.

Proof. We first notice the Postnikov tower on MU can be applied termwise on cosimplicial spectrum $s \mapsto \mathrm{MU}^{\wedge s+1}$ and incudes a filtration on it. (note here this filtration is indeed complete by Example 2.1.7).

Since $\mathrm{Re}_{\mathbb{C}}$ is an exact symmetric monoidal functor, it preserves cofiber sequences. By Theorem 4.4.1 and Corollary 4.4.2, we have an isomorphism

$$\mathrm{Re}_{\mathbb{C}}\left(\mathrm{colim}_{s\leq N}\mathrm{Tot}_s\mathrm{gr}^{[a,b)}\mathrm{MGL}^{\wedge *+1}\right) \cong \mathrm{colim}_{s\leq N}\mathrm{Tot}_s\tau^{\mathrm{B}}_{\geq 2a}\tau^{\mathrm{B}}_{\leq 2b}\mathrm{MU}^{\wedge *+1}. \tag{5.2.6}$$

We still need to calculate the left side to get rid of the colimit. We observe this follows from the descendability of $\mathrm{gr}^{[a,b)}\mathbb{S}_k^* \to \mathrm{gr}^{[a,b)}(\iota_s)_*\mathrm{MGL}^{\wedge *+1}$: Indeed, descendability implies an equivalence

$$\mathrm{gr}^{[a,b)}\mathbb{S}_k^* \simeq \mathrm{Tot}_s\mathrm{gr}^{[a,b)}(\iota_s)_*\mathrm{MGL}^{\wedge *+1} \tag{5.2.7}$$

in $\mathcal{SH}(k) = \mathrm{Mod}_{\mathbb{S}_k}$. Now take $N = \infty$ and by definition of décalage functor, we have the desired result. □

Therefore, we reduce to show:

Lemma 5.2.3 *Let $a \leq b \leq s+1$, denote $\iota_s^* : \Delta^{\leq s} \hookrightarrow \Delta$ the inclusion of simplicial sets and $(\iota_s)_*$ its right adjoint. Then $\mathrm{gr}^{[a,b)}\mathbb{S}_k^* \to \mathrm{gr}^{[a,b)}(\iota_s)_*\mathrm{MGL}^{\wedge *+1}$ is $\mathcal{E}_\infty$-descendable.*

Proof. The unit map $c\mathbb{S}_k \to \mathrm{MGL}^{\wedge *+1}$ induces a map $\mathbb{S}_k \to \mathrm{holim}_{\Delta^{\leq s}}(\iota_s)_*\mathrm{MGL}^{\wedge *+1}$ since $\mathrm{holim}(\iota_s)_*\mathbb{S}_k \cong \mathbb{S}_k$. On the other hand, there is an equivalence

$$\mathrm{holim}_{\Delta^{\leq s}}(\iota_s)_*\mathrm{MGL}^{\wedge *+1} \simeq \mathrm{Tot}_s(\iota_s)_*\mathrm{MGL}^{\wedge *+1} \tag{5.2.8}$$

by definition. We now show $\mathrm{gr}^{[a,b)}\mathbb{S}_k^* \cong \mathrm{Tot}_s \mathrm{gr}^{[a,b)}(\iota_s)_* \mathrm{MGL}^{\wedge *+1}$, which, in light of Proposition 6.8.2, suffices to construct a retraction from $\mathrm{Tot}^s \mathrm{gr}^{[a,b)}(\iota_s)_* \mathrm{MGL}^{\wedge *+1}$ to $\mathrm{gr}^{[a,b)}\mathbb{S}_k^*$.

We show $\mathrm{cofib}\left(\mathrm{gr}^{[a,b)}\mathbb{S}_k^* \to \mathrm{Tot}_s \mathrm{gr}^{[a,b)}(\iota_s)_* \mathrm{MGL}^{\wedge *+1}\right) = 0$. For this, notice that by Theorem 4.2.1 $\mathrm{gr}^0\mathbb{S}_k^* \cong \mathrm{gr}^0\mathrm{MGL}^* \simeq M_{\mathbb{Z}}$. Let $\overline{\mathrm{MGL}}$ be the cofiber of $\mathbb{S}_k \to \mathrm{MGL}$, then it follows that $\overline{\mathrm{MGL}}^1 = \overline{\mathrm{MGL}}$, thus $\left(\overline{\mathrm{MGL}}^{\wedge s+1}\right)^{s+1} = \overline{\mathrm{MGL}}^{\wedge s+1}$, and we have

$$\left(\Omega^s \overline{\mathrm{MGL}}^{\wedge s+1}\right)^{s+1} \cong \Omega^s \overline{\mathrm{MGL}}^{\wedge s+1}. \tag{5.2.9}$$

Hence for $a \leq b \leq s+1$

$$\mathrm{gr}^{[a,b)}\left(\Omega^s \overline{\mathrm{MGL}}^{\wedge s+1}\right)^{\star} = 0. \tag{5.2.10}$$

The following fiber sequence is well known [[Lur17], §4.7.2]:

$$\Omega^s \overline{\mathrm{MGL}}^{\wedge s+1} \to \mathrm{Tot}_s \mathrm{MGL}^{\wedge *+1} \to \mathrm{Tot}_{s-1} \mathrm{MGL}^{\wedge *+1} \tag{5.2.11}$$

and after truncation to $((\iota)_s)_* \mathrm{MGL}^{\wedge *+1}$ we have a cofiber sequence

$$\mathbb{S}_k \simeq \mathrm{Tot}_{s-1}((\iota)_s)_* \mathrm{MGL}^{\wedge *+1} \to \mathrm{Tot}_s((\iota)_s)_* \mathrm{MGL}^{\wedge *+1} \to \Omega^s \overline{\mathrm{MGL}}^{\wedge s+1}. \tag{5.2.12}$$

Then we conclude by taking the associated graded pieces. □

Proof of Theorem 5.3.1. Since a and b in Lemma 5.2.2 are arbitrary, we have an isomorphism of the slice spectral sequences associated to two filtrations $\mathrm{Re}(\mathbb{S}_k^\star)$ and $\mathrm{Dec}^\bullet(\mathrm{MU}^{\wedge *+1})$, where all the odd homotopy groups of $\mathrm{MU}^{\wedge s+1}$ vanish.

On the other hand, the realization functor induces an isomorphism

$$\pi_{n,0}\left(\mathrm{gr}^{[a,b)}\mathbb{S}_k^\star\right)(k) \cong \pi_n\left(\mathrm{Re}\left(\mathrm{gr}^{[a,b)}\mathbb{S}_k^\star\right)\right) \tag{5.2.13}$$

by Theorem 4.4.5, whence the first spectral sequence is just $E(AH)$ by definition. This, after a change of E_2-spectral sequence to E_1-spectral sequence, yields

$$E_r^{p,q}(AH) \cong E_{2r-1}^{2p,q-p}(\mathrm{Dec}(\mathrm{MU}^{\wedge *+1})) \cong E_{2r}^{2p,q-p}(\mathrm{Dec}(\mathrm{MU}^{\wedge *})). \tag{5.2.14}$$

Now by Theorem 2.3.12:

$$E_{2r}^{2p,q-p}(\mathrm{Dec}(\mathrm{MU}^{\wedge *})) \cong E_{2r+1}^{3p+q,-2p}(\mathrm{MU}^{\wedge *}) = E_{2r+1}^{3p+q,2p}(AN) \tag{5.2.15}$$

where the last change of indices is from the Cartan-Eilenberg indexing convention to the one of Bousfield-Kan, which is common for the Adams-Novikov spectral sequence. □

Corollary 5.2.4 *Fix a prime ℓ and the associated Brown-Peterson spectrum $\mathrm{BP}^{(\ell)}$. Then there is an isomorphism of the ℓ-local Adams-Novikov spectral sequence*

$$E_2^{s,t}(AN)_\ell = \mathrm{Ext}^{s,t}_{\mathrm{BP}_*^{(\ell)}(\mathrm{BP}^{(\ell)})}\left(\mathrm{BP}_*^{(\ell)}, \mathrm{BP}_*^{(\ell)}\right) \Longrightarrow \pi_{t-s}\mathbb{S} \otimes \mathbb{Z}_{(\ell)} \tag{5.2.16}$$

and the ℓ-local motivic Atiyah-Hirzebruch spectral sequence

$$E_1^{p,q}(AH)_\ell = \pi_{-p-q,0}(\mathrm{gr}^{-q}\mathbb{S}_k^\star)(k) \otimes \mathbb{Z}_{(\ell)} \Longrightarrow \pi_{-p-q,0}(\mathbb{S}_k)(k) \otimes \mathbb{Z}_{(\ell)}. \tag{5.2.17}$$

induced by the complex Betti realization.

Proof. As of Definition 3.4.14, $\mathrm{BP}^{(\ell)}_{\mathrm{mot}}$ is a direct summand of $\mathrm{MGL}_{(\ell)}$ so it lies in $\mathcal{SH}^{\mathrm{eff}}(k)$. Moreover, by Theorem 4.2.1 and Corollary 4.2.7 we have

$$\mathrm{gr}^0\mathbb{S}^*_k \otimes \mathbb{Z}_{(\ell)} \simeq \mathrm{gr}^0\mathrm{MGL}^* \otimes \mathbb{Z}_{(\ell)} \simeq \mathrm{gr}^0\mathrm{BP}^{(l)}_{\mathrm{mot}} \tag{5.2.18}$$

and therefore the argument in Lemma 5.2.3 can be applied and the unit map $\mathbb{S}^*_k \otimes \mathbb{Z}_{(\ell)} \to \mathrm{BP}^{(\ell)}_{\mathrm{mot}}$ induces a descent on grading pieces. Finally by Corollary 4.4.3 we have

$$\mathrm{Re}\Big(\mathrm{gr}^{[a,b)}\big(\mathbb{S}_k \otimes \mathbb{Z}_{(\ell)}\big)^{\star}\Big) \cong \mathrm{gr}^{[2a,2b)}\mathrm{Dec}^{\bullet}\Big(\big(\mathrm{BP}^{(\ell)}\big)^{\wedge *+1}\Big) \tag{5.2.19}$$

and the rest is analogue. □

Remark 5.2.5 This isomorphism of spectral sequences has a conceptual interpretation: In [Pst23] it was shown that

$$\mathcal{SH}^{\mathrm{cell}}(\mathbb{C})^{\wedge}_p \simeq \mathrm{Syn}^{\wedge}_p \tag{5.2.20}$$

where Syn represents the category of synthetic spectra as constructed in [[Pst23], §4.1], which is a way to encode the Adams-Novikov spectral sequence using a one-parameter deformation of $\mathcal{SH}$. In [Ghe+22], another attempt of categorification of the Adams-Novikov spectral sequence is to introduce $\Gamma_*\mathbb{1}$-modules of filtered spectra: this shares the same spirit as décalage introduced by Antieau later (see [[Ghe+22], Remark 3.7]). Nevertheless, the two constructions turned out to be equivalent [[Pst25], Theorem 7.5]: they are both related to the concept of even filtrations, which is a purely topological construction.

Since MGL is cellular, this equivalence reveals that over $\mathbb{C}$, the behavior of MGL-modules should be purely topological under some mild finiteness conditions. In fact, the étale case suggests that more should be true over arbitrary algebraic closed fields, though we don't know how to precisely state that, since cellularity is not closed under infinite limits, in particular, completions.

5.3 A spectral sequence of étale cobordism

We now turn our attention to the ℓ-adic case. We fix an algebraically closed field of characteristic p and a prime $\ell \neq p$. In this section we may replace $\mathbb{S}_k$ by $\mathbb{S}_k[1/p]$ in order to use the Hopkins-Morel theorem Theorem 4.2.1.

Unlike the characteristic zero case, the completed spectra does not always behave well under realization since it is an infinite limit. However, the main theorem of [Elm+22] gives us a possible approach.

Recall we have built our motivic spaces out of Nisnevich sheaves of smooth schemes. Since étale site is finer that Nisnevich site, there is not obstruction to define an étale version of stable motivic homotopy category, of which we denote $\mathcal{SH}_{\text{ét}}(S)$. This category turns out to be a non-full localization of $\mathcal{SH}(S)$. The identity functor $\mathrm{id} : \mathcal{S}\mathrm{m}_S \to \mathcal{S}\mathrm{m}_S$ induces a geometric morphism of ∞-topoi

$$\varepsilon_* : \mathrm{Shv}^{\wedge}_{\text{ét}}(\mathcal{S}\mathrm{m}_S) \to \mathrm{Shv}_{\mathrm{Nis}}(\mathcal{S}\mathrm{m}_S) \tag{5.3.1}$$

with a left adjoint ε^*. This adjunction descends to

$$\varepsilon^* : \mathcal{SH}(S) \rightleftarrows \mathcal{SH}_{\text{ét}}(S) : \varepsilon_* \tag{5.3.2}$$

By construction, there is a canonical isomorphism of motivic cohomology:

$$H^{0,1}(\mathrm{Spec}\ k; \mathbb{Z}/n) \simeq \mu_n(k). \tag{5.3.3}$$

Now let ζ be a primitive n-th roots of unity in k, and let β_n be the associated element of $H^{0,1}(\mathrm{Spec}\ k; \mathbb{Z}/n)$. As in [[Hoy15], 8.14], the spectral sequence

$$H^{p+2t,q+t}(k; \mathbb{Z}/n) \otimes L_t[1/p] \Longrightarrow \mathrm{MGL}^{p,q}(k)[1/p] \tag{5.3.4}$$

sends β_n to an element in $\mathrm{MGL}^{0,1}(k; \mathbb{Z}/n)$, which we call motivic Bott element. As explained in [Elm+22], the element β_{ℓ^v} actually lives in $(\mathrm{MGL}/\ell^v)^{0,N}(k)$ for some N, and the formal inversion with respect to β_{ℓ^v} is independent on the choice of the root of unity ζ.

Proposition 5.3.1 *[[Elm+22], Theorem 6.26] For any $v \geq 1$ The unit of the adjunction (5.3.2) induces an equivalence of spectra*

$$\mathrm{MGL}/\ell^v[\beta_{\ell^v}^{-1}] \xrightarrow{\simeq} \mathrm{MGL}^{\text{ét}}/\ell^v \tag{5.3.5}$$

where $\mathrm{MGL}^{\text{ét}} := \varepsilon_*\varepsilon^*(\mathrm{MGL})$ the étale localization of algebraic cobordism.

Remark 5.3.2 In [Qui07] this étale localization of algebraic cobordism is identified with his étale cobordism spectrum after étale realization. This explains the title of this section.

From this proposition and Theorem 5.1.8, an argument of slice spectral sequences yields:

Theorem 5.3.3 *[[Elm+22], Proposition 7.11, [Qui07], Theorem 64] There is an isomorphism of graded abelian groups induced by étale realization:*

$$\left(\bigoplus_{p,q} \mathrm{MGL}^{p,q}(k) \otimes \mathbb{Z}_\ell\right)[\beta^{-1}] \cong \bigoplus_p (\mathrm{MU}^{\wedge}_\ell)^p[\beta^{-1}] \tag{5.3.6}$$

where β is the collection of β_{ℓ^v} for all $v \geq 1$.

More generally, this isomorphism works for all Landweber exact theories as pointed out by [[Elm+22], Proposition 7.12]. This allows us to prove the following theorem:

Theorem 5.3.4 *Replace all spectra with Bott inverted version. Let $\mathrm{MU}^{\wedge}_\ell$ be the ℓ-adic completion of complex cobordism spectrum. The Adams-Novikov spectral sequence*

$$E_2^{p,q}(AN)^{\wedge}_\ell = \mathrm{Ext}^{s,t}_{\mathrm{MU}^{\wedge}_{\ell,*}(\mathrm{MU}^{\wedge}_\ell)}\left(\mathrm{MU}^{\wedge}_{\ell,*}, \mathrm{MU}^{\wedge}_{\ell,*}\right) \Longrightarrow (\pi_{t-s}\mathbb{S})^{\wedge}_\ell \tag{5.3.7}$$

converges and is isomorphic to the ℓ-complete Atiyah-Hirzebruch spectral sequence

$$E_1^{s,t}(AH)^{\wedge}_\ell = \pi_{-s-t,0}(\mathrm{gr}^{-t}\mathbb{S}^*_k[1/p])(k) \otimes \mathbb{Z}_\ell \Longrightarrow \pi_{-s-t,0}(\mathbb{S}_k[1/p])(k) \otimes \mathbb{Z}_\ell \tag{5.3.8}$$

with isomorphisms induced by the étale realization functor. In other words, there is an isomorphism

$$\left(\pi_{s+t,0}(\mathbb{S}_k[1/p])(k)\right)^\wedge_\ell[\beta^{-1}] \cong \left(\pi_{s+t}\mathbb{S}\right)^\wedge_\ell[\beta^{-1}] \tag{5.3.9}$$

Proof. By Theorem 5.1.8 the second spectral sequence is convergent, hence it suffices to show the isomorphism as stated.

The argument in Lemma 5.2.3 for

$$\mathbb{S}_k \otimes \mathbb{Z}_\ell \to \mathrm{MGL} \otimes \mathbb{Z}_\ell \tag{5.3.10}$$

gives us the descent object. By Theorem 5.3.3, an isomorphism

$$\pi_{*,*}(\mathrm{gr}^s\mathbb{S}_k^*[1/p])(k)[\beta^{-1}] \otimes \mathbb{Z}_\ell \cong \pi_*\left(\mathrm{gr}^{2s}\mathrm{Dec}^\bullet\left((\mathrm{MU}_\ell^\wedge)^{\wedge *+1}\right)\right)[\beta^{-1}]. \tag{5.3.11}$$

The two filtrations agree globally after inverting Bott element, even though we can't say anything else over arbitrary section! Now the usual décalage trick can be applied. □

Remark 5.3.5 After finishing this thesis, we notice that our result has a partial overlapping with the main result in [[BBX25], Theorem 8.3], where they compared the motivic stable stem over any field with the one over $\mathbb{C}$. Concretely they proved the following isomorphism of derived complete modules of $K_*^{MW}(\mathbb{C})^\wedge_{\ell,\eta}$:

$$\pi_{*,*}\left(\mathbb{S}^\wedge_{\ell,\eta}\right)(k) \simeq K_*^{MW}(k)^\wedge_{\ell,\eta} \,\hat{\otimes}\, \pi_{*,*}\left(\mathbb{S}^\wedge_{\ell,\eta}\right)(\mathbb{C}) \tag{5.3.12}$$

where they only assumed k to be Tate-orientable, i.e. contains μ_{ℓ^n} for all n.

6 Appendix: Higher category theory

We will collect and quickly go through most category theoretical languages used in this thesis, especially the ∞-category, which might not be familiar to some readers.

6.1 Definition of $(\infty, 1)$-category

There're many approaches to introduce higher and ∞-categories, perhaps the best one with a balance of combinatorial background and transparency is the theory of quasicategories (resp. weak Kan complexes).

Definition 6.1.1 The simplex category Δ is the category of non-empty finite totally ordered sets and order-preserving maps between them. Typically every object in Δ is isomorphic to $[n] = \{0 \leq 1 \leq 2 \leq ... \leq n\}$ for some n and isomorphisms are unique if exists.

Definition 6.1.2 Let $\mathcal{C}$ be a category, a simplicial object in $\mathcal{C}$ is a functor $\Delta^{\mathrm{op}} \to \mathcal{C}$. Similarly a cosimplicial object is a functor $\Delta \to \mathcal{C}$. In particular, a simplicial set is a functor $\Delta^{\mathrm{op}} \to \mathrm{Set}$. The functor category of all simplicial objects in $\mathcal{C}$ is denoted by $\mathcal{C}_\Delta$.

Definition 6.1.3 The i-th face map $d_i : [n-1] \to [n]$ is the unique morphism whose image doesn't contain i, and the i-th degeneracy maps $s_i : [n+1] \to [n]$ is the unique morphism whose image hits i twice.

Remark 6.1.4 It's easy to see every morphism in Δ can be written in a composition of face maps and degeneracy maps. In light of this observation, we may identify a simplicial object S in $\mathcal{C}$ as following data:
1. for each $n \geq 0$ an object S_n and
2. for each i the (pullback) $d_i : S_{n-1} \to S_n$ (of) face maps and $s_i : S_{n+1} \to S_n$ of degeneracy maps.

Definition 6.1.5 For every natural number n, the n-simplex Δ^n is defined to be the simplicial set represented by the object $[n]$ as with the Yoneda lemma. For each n-complex where $n \geq 1$ and let $0 \leq k \leq n$, we define the k-th horn of Δ^n to be the simplicial set $\Lambda^n_k \subset \Delta^n$ as the union of all faces except k-th in Δ^n.

Remark 6.1.6 For every topological space X, we may associate the singular simplicial complex $\mathrm{Sing}(X)$ to it as in algebraic topology. This defines a functor $\mathrm{Top} \to \mathcal{S}\mathrm{et}_\Delta$ and it admits a left adjoint, which assigns a geometric realization $|S|$ to each simplicial set S, which is left Kan extended from the functor $[n] \mapsto \Delta^n$.

The ability to extend maps out of horns to maps out of simplices is a criterion for homotopy coherence.

Definition 6.1.7 Let K be a simplicial set, we say:
1. K is a Kan complex if for each $n \geq 1$ and $0 \leq k \leq n$, every morphism of simplicial sets $\Lambda^n_k \to K$ can be extended to a morphism $\Delta^n \to X$.
2. K is a quasicategory if the above condition holds just for $0 < k < n$.

Example 6.1.8
1. The singular complex $\mathrm{Sing}(X)$ of a topological space is a Kan complex X.
2. Let $\mathcal{C}$ be a small 1-category, the nerve of $\mathcal{C}$ is defined as the following simplicial set: for each $n \geq 0$, $\mathrm{N}(\mathcal{C})_n$ is the set of functors $[n] \to \mathcal{C}$, i.e. the chain
$$C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} C_2 \to \ldots \xrightarrow{f_n} C_n \tag{6.1.1}$$
the pull back of j-th face map sends this to
$$C_0 \xrightarrow{f_1} C_1 \to \ldots C_{j-1} \xrightarrow{f_{j+1} \circ f_j} C_{j+1} \to \ldots \xrightarrow{f_n} C_n \tag{6.1.2}$$
and the pull back of j-the degeneracy map sends objects in $\mathrm{N}(\mathcal{C})_n$ to

$$C_0 \xrightarrow{f_1} C_1 \to \dots \xrightarrow{f_j} C_j \xrightarrow{\mathrm{id}} C_j \xrightarrow{f_j} \dots \xrightarrow{f_n} C_n. \tag{6.1.3}$$

The nerve of a category is a quasicategory, but in general not a Kan complex. In fact, only the nerve of a groupoid is a Kan complex.

Proposition 6.1.9 *[[Lur09], Proposition 1.1.2.2] Let K be a simplicial set, then K is isomorphic to the nerve of a category $\mathcal{C}$ if and only if K is a quasicategory and the extension in Definition 6.1.7 is* unique. *Moreover, the functor* $\mathrm{N} : \mathrm{Cat} \to \mathcal{S}\mathrm{et}_\Delta$ *is fully faithful.*

The proposition above tells us that the theory of quasicategories is indeed an extension of classical category theory. From now on, we will simply refer ∞-category to quasicategory and identify the nerve of a 1-category with itself.

Definition 6.1.10

1. Given an ∞-category $\mathcal{C}$, we say a functor $\Delta^0 \to \mathcal{C}$ is an object of $\mathcal{C}$ and $\Delta^1 \to \mathcal{C}$ a morphism of $\mathcal{C}$. A morphism $f : \Delta^1 \to \mathcal{C}$ has source $d_0(f)$ and target $d_1(f)$. If X is an object in $\mathcal{C}$, then we refer $\mathrm{id}_X := s_0(X)$ to the identity morphism of X.
2. [Segal axiom] Given two morphisms $f, g : \Delta^1 \to \mathcal{C}$ with $d_1(f) = d_0(g)$, the composition $g \circ f : \Delta^1 \to \mathcal{C}$ is exhibited as the extension of $\Lambda^2_1 \to \mathcal{C}$ to a unique commutative triangle $\Delta^2 \to \mathcal{C}$, where f, g are 0th and 2nd faces respectively.
3. For any two ∞-categories $\mathcal{C}$ and $\mathcal{D}$, we say $F : \mathcal{C} \to \mathcal{D}$ is a functor of ∞-categories if it is a morphism of underlying simplicial sets of $\mathcal{C}$ and $\mathcal{D}$. In general if $\mathcal{C}$ is a simplicial set and $\mathcal{D}$ is an ∞-category, then $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is also an ∞-category.

Remark 6.1.11 The reader may ask in which purpose we have introduced Kan complex as a more strict notation. In fact, one can show the geometric realization of a Kan complex is related to a CW-complex, thus Kan complexes share the same homotopy properties like our usual understanding of a "space".

A moderner name for Kan complexes is anima, this was introduced recently by Clausen and Scholze. Informally, the reader can understand an anima as a generalization of a set, or a space with only homotopy type captured, forgetting the underlying geometry.

In order to prevent being confused by concrete models and constructions, especially the model of simplicial sets above, we define the following objects in an axiomatic way:

Definition 6.1.12 There exists two ∞-categories An and Cat_∞, called the ∞-category is small animae and small ∞-categories, such that:

1. $\mathrm{An} \hookrightarrow \mathrm{Cat}_\infty$ is fully faithful.

2. For any object $c \in \mathrm{Cat}_\infty$, we can associate an up to equivalence unique small ∞-category $\mathcal{C}$. The same holds for An.

We are ready to define the homotopy category of an ∞-category.

Proposition 6.1.13 *[[Lur09], Proposition 1.2.3.1] The functor $N : \mathrm{Cat} \to \mathrm{Cat}_\infty$ has a left adjoint $\mathrm{h} : \mathrm{Cat}_\infty \to \mathrm{Cat}$ which sends an ∞-category $\mathcal{C}$ to its homotopy category $\mathrm{h}\mathcal{C}$.*

Remark 6.1.14 Thanks to the extension property of quasicategories, we may have a good description of the homotopy category of an ∞-category, using our usual understanding of homotopy between maps. The reader may refer to [[Lur09], §1.2.3].

Definition 6.1.15

1. A morphism $f : \Delta^1 \to \mathcal{C}$ in a ∞-category $\mathcal{C}$ is said to be an equivalence if it is an isomorphism in $\mathrm{h}\ \mathcal{C}$.
2. Let $\mathcal{C}$ be an ∞-category, let $\mathcal{C}^\simeq \subset \mathcal{C}$ be the largest simplicial subset such that all morphisms in $\mathcal{C}^\simeq$ are equivalences. Then $\mathcal{C}^\simeq$ is an anima and is called the core of $\mathcal{C}$. For any other anima K, the functor $K \to \mathcal{C}$ must factor through $\mathcal{C}^\simeq$.

The Hom-set between two objects in ordinary category now can be extended to a homotopy object, called the mapping space in ∞-category.

Definition 6.1.16 Let S be a simplicial set, $x, y \in S$, the mapping space $\mathrm{Map}_S(x, y)$ between x and y is just the space $\mathrm{Map}_{\mathrm{h}\ S}(x, y) \in \mathcal{H}$ in homotopy category, where $\mathrm{h}\ S$ denote the homotopy category of S regarded as a $\mathcal{H}$-enriched category.

We do not need to worry too much about the technical details of $\mathcal{H}$-enrichment as discussed in [Lur09]. What really matters is the following theorem:

Proposition 6.1.17 *[[Lur09], Proposition 1.2.2.3] Let $\mathcal{C}$ be an ∞-category, then for any two objects $x, y \in \mathcal{C}$, the mapping space $\mathrm{Map}_{\mathcal{C}}(x, y)$ is an anima, called the Hom anima of x, y in $\mathcal{C}$.*

Using straightening and unstraightening as in [[Lur09], §3.2], one can show that this extends to a functor

$$\mathrm{Hom}_{\mathcal{C}}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{An} \tag{6.1.4}$$

where $\mathcal{C}$ is an ∞-category and An is the ∞-category of all (small) animae, which is defined in Definition 6.1.12.

Definition 6.1.18 For any two objects $c, d \in \mathrm{Cat}_\infty$ corresponding to small ∞-categories $\mathcal{C}$ and $\mathcal{D}$, there is an equivalence of animae

$$\mathrm{Hom}_{\mathrm{Cat}_\infty}(c,d) \overset{\simeq}{\to} \mathrm{Map}_{\mathrm{Set}_\Delta}(\mathcal{C},\mathcal{D}). \tag{6.1.5}$$

In particular, we can also define the functor category from $\mathcal{C}$ to $\mathcal{D}$ as the ∞-category

$$\mathrm{Fun}(\mathcal{C},\mathcal{D}) := \mathrm{Hom}_{\mathrm{Cat}_\infty}(c,d). \tag{6.1.6}$$

It's convenient to have initial and terminal objects regarding to all ∞-categories.

Definition 6.1.19

1. The ∞-category $* = [0]$ is terminal in the sense that for any ∞-category $\mathcal{C}$, there is a unique(up to equivalence) functor $\mathcal{C} \to *$. The ∞-category $\emptyset$ is strict initial in the sense that there is a unique functor $\emptyset \to \mathcal{C}$, and every functor $\mathcal{C} \to \emptyset$ is necessarily an equivalence.
2. An ∞-category is contractible if $\mathcal{C} \to *$ is an equivalence.
3. Let $\mathcal{C}$ be an ∞-category and $x \in \mathcal{C}$. x is said to be an initial object if for any $y \in \mathcal{C}$, the mapping space $\mathrm{Map}_{\mathcal{C}}(x,y)$ is contractible. Analogously, x is a final object if $\mathrm{Map}_{\mathcal{C}}(y,x)$ is contractible for any $y \in \mathcal{C}$.

We can use the above definition to talk about limits and colimits in ∞-category as in the usual 1 -category. Be careful the uniqueness of these objects are all up to the contractibility of mapping spaces.

Finally we introduce two constructions of new ∞-categories: subcategories spanned by morphisms and localizations at some morphisms.

Definition 6.1.20

1. A monomorphism between two ∞-categories $\mathcal{C},\mathcal{D}$ is a functor $F : \mathcal{C} \to \mathcal{D}$ such that

$$\begin{tikzcd} C \arrow[r, "="] \arrow[d, "="'] & C \arrow[d, "F"] \\ C \arrow[r, "F"] & D \end{tikzcd}$$

is a pullback square.

2. Let $\mathcal{C}$ be an ∞-category, a collection of morphisms in $\mathcal{C}$ is a monomorphism $M \hookrightarrow \mathrm{Map}(\Delta^1,\mathcal{C})$ where M is another small ∞-category. We say the collection is closed under composition if for any $f : \Delta^1 \to \mathcal{C}$ in M between $x, y \in \mathcal{C}$, the morphisms $\mathrm{id}_x, \mathrm{id}_y$ are also in M. If $g \in M$, then $g \circ f : \Delta^1 \to \mathcal{C}$ is also in M.

Definition 6.1.21

1. We refer $\langle M\rangle_{\mathcal{C}}$ together with a functor $i_M : \langle M\rangle_{\mathcal{C}} \to \mathcal{C}$ to the subcategory spanned by collection of morphisms M in $\mathcal{C}$, where M is closed under composition, if the following conditions hold:
   1. The induced functor $(i_M)_* : \mathrm{Fun}(\Delta^1, \langle M\rangle_{\mathcal{C}}) \to \mathrm{Fun}(\Delta^1,\mathcal{C})$ factors through M.

2. For any ∞-category $\mathcal{D}$, we have a pullback square

$$\begin{tikzcd} \mathrm{Fun}(\mathcal{D}, \langle M\rangle_{\mathcal{C}}) \arrow[r] \arrow[d] & \mathrm{Fun}(\mathcal{D}, \mathcal{C}) \arrow[d] \\ \mathrm{Fun}(\mathrm{Fun}(\Delta^1, \mathcal{D}), M) \arrow[r] & \mathrm{Fun}(\mathrm{Fun}(\Delta^1, \mathcal{D}), \mathrm{Fun}(\Delta^1, \mathcal{C})). \end{tikzcd}$$

2. Let $\Gamma \hookrightarrow \mathcal{C}^{\simeq}$ be an embedding of ∞-categories. Let $\mathrm{ev}_1 : \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}^{\simeq} \times \mathcal{C}^{\simeq}$ be the map extracting source and target objects of a morphism, Set M_Γ to be the pullback of ev_1 and inclusion $\Gamma \times \Gamma \hookrightarrow \mathcal{C}^{\simeq} \times \mathcal{C}^{\simeq}$. We say $\langle M_\Gamma\rangle_{\mathcal{C}}$ is the full subcategory spanned by objects in Γ in $\mathcal{C}$.

Another construction is localization, where we invert certain family of morphisms.

Definition 6.1.22
1. Let $\mathcal{C}$ be an ∞-category, let $W \hookrightarrow \mathrm{Fun}(\Delta^1, \mathcal{C})$ be a collection of morphisms. An ∞-category $\mathcal{C}[W^{-1}]$ together with a functor $l : \mathcal{C} \to \mathcal{C}[W^{-1}]$ is called the Dwyer-Kan localization of $\mathcal{C}$ at the morphisms in W if the following conditions hold:
1. The functor l sends the morphisms in W into equivalences in $\mathcal{C}[W^{-1}]$.
2. For any ∞-category $\mathcal{D}$ we have a pullback square

$$\begin{tikzcd} \mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{D}) \arrow[r] \arrow[d] & \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \arrow[d] \\ \mathrm{Fun}(W, \mathrm{Iso}(\mathcal{D})^{\simeq}) \arrow[r] & \mathrm{Fun}(W, \mathrm{Fun}(\Delta^1, \mathcal{D})). \end{tikzcd}$$

where $\mathrm{Iso}(\mathcal{D})$ is the subcategory spanned by all equivalences in $\mathcal{D}$.
2. In case W is the collection of all morphisms, we call the localization the geometric realization of $\mathcal{C}$ and denoted by $|\mathcal{C}|$, it is an anima by definition.

The localization of Top with respect to all weak equivalences is equivalent to An. This is exactly the content of the homotopy hypothesis due to Grothendieck.

6.2 Stable ∞-category

We introduce the notion of stable ∞-categories. We show the homotopy category of a stable ∞-category is always a triangulated category. We will also discuss t-structures on these homotopy categories.

Definition 6.2.1 An ∞-category $\mathcal{C}$ is pointed if there is a object 0 serving as the initial and terminal objects at the same time.

Remark 6.2.2 There is a canonical way to equip a minimal pointed structure $\mathcal{C}_*$ on an ∞-category $\mathcal{C}$ with a terminal object $*$ by the pullback square

$$\begin{tikzcd}
\mathcal{C}_* \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1],\mathcal{C}) \arrow[d, "\mathrm{ev}_0"'] \\
* \arrow[r, "*"] & \mathcal{C}
\end{tikzcd}$$

Readers that have knowledge of algebraic topology may be familiar with following constructions:

Definition 6.2.3 Let $\mathcal{C}$ be a pointed ∞-category, a null sequence(or triangle) in $\mathcal{C}$ is a commutative square

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"'] \\
0 \arrow[r] & Z
\end{tikzcd}$$

Such a sequence is called:

1. a fiber sequence if it is a pullback square in $\mathcal{C}$ and we write $X \simeq \mathrm{fib}(g)$,
2. a cofiber sequence if it is a pushout square in $\mathcal{C}$ and we write $Z \simeq \mathrm{cofib}(f)$.

There are some special fibers and cofibers which deserve a name:

Definition 6.2.4 Let $\mathcal{C}$ be a pointed ∞-category,

1. suppose $\mathcal{C}$ admits fibers, then the fiber of the unique map $0 \to X$ is the loop object ΩX of X, this defines a functor $\Omega : \mathcal{C} \to \mathcal{C}$.
2. suppose $\mathcal{C}$ admits cofibers, then the cofiber of the unique map $X \to 0$ is the suspension object ΣX of X, this defines a functor $\Sigma : \mathcal{C} \to \mathcal{C}$.

Remark 6.2.5 We are not following the strict treatment of ∞-category in the sense of §6.1 and [Lur09]. Strictly speaking, the uniqueness here is up to contractibility of mapping spaces. A much more formal definition of theses two functors can be found in [Lur17].

Proposition 6.2.6 *Let $\mathcal{C}$ be a pointed ∞-category that admits fibers and cofibers, then the suspension functor $\Sigma : \mathcal{C} \to \mathcal{C}$ is left adjoint to $\Omega : \mathcal{C} \to \mathcal{C}$.*

We are ready to define stable ∞-categories.

Definition 6.2.7 Let $\mathcal{C}$ be a pointed ∞-category, then $\mathcal{C}$ is a stable ∞-category if $\mathcal{C}$ admits fibers and cofibers, and a null sequence is a fiber sequence iff it is a cofiber sequence.

One can check that there are following equivalent definitions of stable ∞-categories.

Proposition 6.2.8 *For a pointed ∞-category $\mathcal{C}$ the following are equivalent:*

1. *$\mathcal{C}$ is stable.*
2. *$\mathcal{C}$ admits finite limits and colimits, and a commutative square is a pullback square iff it is a pushout square, i.e. an exact square.*
3. *$\mathcal{C}$ admits fibers and the loop functor $\Omega : \mathcal{C} \to \mathcal{C}$ is an equivalence.*
4. *$\mathcal{C}$ admits cofibers and the suspension functor $\Sigma : \mathcal{C} \to \mathcal{C}$ is an equivalence.*

Proof. see e.g. [[Cno25], Theorem 3.2.7]. □

An immediate consequence of Proposition 6.2.8 is:

Corollary 6.2.9 *The homotopy category of every stable ∞-category is additive.*

We now install a triangulated structure on the underlying homotopy category $\mathrm{h}\,\mathcal{C}$ of a stable ∞-category $\mathcal{C}$. For this, let us briefly review the theory of triangulated category and t-structures. Let K be an additive category with an endofunctor $[1] : K \to K$, called the suspension functor. A triangle in an additive category K is a tuple (X, Y, Z, u, v, w) where $u : X \to Y, v : Y \to Z, w : Z \to X[1]$.

K is said to be a triangulated category if there's a collection of distinguished triangles (X, Y, Z, u, v, w) that fulfils following properties:

TR0. Every triangle that is isomorphic to a distinguished triangle is automatically distinguished.

TR1. The triangle $(X, X, 0, 1_X, 0, 0)$ is distinguished.

TR2. Every morphism $u : X \to Y$ embeds into a distinguished triangle (X, Y, Z, u, v, w).

TR3. The triangle (X, Y, Z, u, v, w) is a distinguished triangle iff $(Y, Z, X[1], v, w, -u[1])$ is.

TR4. Every commutative square

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ \downarrow & & \downarrow \\ X' & \xrightarrow{u'} & Y' \end{array}$$

embeds into a morphism of distinguished triangles.

TR5. (octahedral axiom) Given three distinguished triangles

$$\begin{aligned} X &\xrightarrow{u} Y \to Z' \to X[1] \\ Y &\xrightarrow{v} Z \to X' \to Y[1] \\ X &\xrightarrow{vu} Z \to Y' \to X[1] \end{aligned} \tag{6.2.1}$$

there's a distinguished triangle

$$Z' \xrightarrow{w} Y' \to X' \to Z'[1] \tag{6.2.2}$$

which completes the octahedral diagram.

$$\begin{tikzcd}
X \arrow[rr, "u"] \arrow[rrrr, bend left, "vu"] & & Y \arrow[rr, "v"] \arrow[dl] & & Z \arrow[dl] \arrow[dll, bend left] \\
& Z' \arrow[ul] \arrow[dr] & & X' \arrow[ul] \arrow[ll, "w"] & \\
& & Y' \arrow[ur] \arrow[ull, bend left] & &
\end{tikzcd}$$

We can define t-structure on a triangulated category: Let $K_{\geq 0}$ and $K_{\leq 0}$ be two strict full subcategories of K. The pair $(K_{\geq 0}, K_{\leq 0})$ is a t-structure on K if

1. for $X \in K_{\leq 0}$ and $Y \in K_{\geq 0}$, $\mathrm{Hom}_K(X, Y[1]) = 0$,
2. $K_{\geq 0}[1] \subset K_{\geq 0}$ and $K_{\leq 0}[-1] \subset K_{\leq 0}$,
3. for any $X \in K$ there's a distinguished triangle

$$Y \to X \to Z \tag{6.2.3}$$

with $Y \in K_{\leq 0}$ and $Z \in K_{\geq 0}[1]$.

The heart of a t-structure $(K_{\geq 0}, K_{\leq 0})$ on K is the full subcategory $K^{\heartsuit} = K_{\geq 0} \cap K_{\leq 0}$ and it is an abelian category.

By playing a little bit with the axioms, we see the inclusion $K_{\leq n} := K_{\leq 0}[n] \hookrightarrow K$ has a right adjoint $\tau_{\leq n} : K \to K_{\leq n}$, similarly, the inclusion $K_{\geq n} := K_{\geq 0}[n] \hookrightarrow K$ has a left adjoint, which we write $\tau_{\geq n} : K \to K_{\geq n}$.

Definition 6.2.10 Let $\mathcal{C}$ be a pointed ∞-category admits cofibers, a triangle (X, Y, Z, f, g, h) in $\mathrm{h}\mathcal{C}$ is said to be distinguished if there exists a diagram in $\mathcal{C}$

$$\begin{tikzcd}
X \arrow[r, "\tilde{f}"] \arrow[d] & Y \arrow[r] \arrow[d, "\tilde{g}"'] & 0 \arrow[d] \\
0 \arrow[r] & Z \arrow[r, "\tilde{h}"] & W
\end{tikzcd}$$

where $\tilde{f}$ and $\tilde{g}$ represents f and g, such that both squares are pushout diagrams in $\mathcal{C}$ and the map $h : Z \to \Sigma X$ is the composition of $\tilde{h}$ and the isomorphism $W \simeq \Sigma X$ determined by outer rectangle.

In a stable ∞-category there's a even simpler characterization.

Definition 6.2.11 For a stable ∞-category $\mathcal{C}$ the general shifting functor $[n] : \mathcal{C} \to \mathcal{C}$ is defined by

$$[n] = \begin{cases} \Sigma^n & \text{if } n \geq 0 \\ \Omega^{-n} & \text{if } n \leq 0 \end{cases} \tag{6.2.4}$$

Note that $[m] \circ [n] \simeq [m+n]$ and $[0] \simeq \mathrm{id}_{\mathcal{C}}$.

Definition 6.2.12 Let $X, Y, Z \in \mathcal{C}$ a stable ∞-category, we say the sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is exact iff

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"'] \\ 0 \arrow[r] & Z \end{tikzcd}$$

is an exact square.

Since $[n]$ is an equivalence, it preserves finite limits and finite colimits, thus also exact squares. By induction and pasting law of pullback and pushout [[Lur09], Lemma 4.4.2.1] we have the following proposition.

Proposition 6.2.13 *If $X \xrightarrow{f} Y \xrightarrow{g} Z$ is an exact sequence in a stable ∞-category $\mathcal{C}$, then:*
1. $Y \xrightarrow{g} Z \to X[1]$ and $Z[-1] \to X \xrightarrow{f} Y$ are also exact.
2. For every integer n, the sequence $X[n] \xrightarrow{f[n]} Y[n] \xrightarrow{g[n]} Z[n]$ is exact.

Proposition 6.2.14 *For $\mathcal{C}$ a stable ∞-category a triangle (X, Y, Z, f, g, h) is distinguished iff there are morphisms $\tilde{f}, \tilde{g}$ in $\mathcal{C}$ representing f, g such that $X \xrightarrow{\tilde{f}} Y \xrightarrow{\tilde{g}} Z$ is an exact sequence and the map $\tilde{h} : Z \to X[1]$ from pasting law [[Lur09], Lemma 4.4.2.1] represents h.*

Theorem 6.2.15 *Let $\mathcal{C}$ be a stable ∞-category, the shifting functor together with distinguished triangles defined in Definition 6.2.12 forms a triangulated structure on $\mathrm{h}\mathcal{C}$.*

Proof. By Corollary 6.2.9 $\mathrm{h}\mathcal{C}$ is an additive category. It suffices to verify axioms of triangulated category.

TR0 and TR1 is obvious.

TR2 follows from the fact that if $\mathcal{C}$ is a stable category, then $\mathcal{C}$ admits cofibers. Pick $\tilde{f}$ representing $f : X \to Y$ in $\mathcal{C}$, then $\left(X, Y, \mathrm{cofib}\left(\tilde{f}\right), f, g, h\right)$ is distinguished where g is the homotopy class of $\tilde{g} : Y \to \mathrm{cofib}\left(\tilde{f}\right)$ in $\mathcal{C}$ and h is canonical.

TR3 is exactly Proposition 6.2.13.

TR4. Suppose we have a diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

representing the diagram in $\mathrm{h}\mathcal{C}$, this can be extended to a diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[r] \arrow[d] & \mathrm{cofib}(f) \arrow[r] \arrow[d] & X[1] \arrow[d] \\
X' \arrow[r, "f'"] & Y' \arrow[r] & \mathrm{cofib}(f') \arrow[r] & X'[1]
\end{tikzcd}$$

since

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\
0 \arrow[r] & \mathrm{cofib}(f)
\end{tikzcd}$$

is a pushout diagram.

TR5. In view of the fact that every two cofibers are uniquely determined up to isomorphism, it suffices to construct a cofiber sequence fits into the stronger commutative diagram in $\mathcal{C}$, which leads to

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[r, "g"] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Z' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & X[1] \arrow[r] \arrow[d] & 0 \arrow[d] \\
 & 0 \arrow[r] & X' \arrow[r] & Y[1] \arrow[r] & Z'[1]
\end{tikzcd}$$

The sequence $Z' \to Y' \to X'$ is exact by repeated use of the pasting law. □

Remark 6.2.16 The notion of stability of an ∞-category is purely intrinsic: we do not need to equip extra structures on it. So is the triangulated structure on its homotopy category. In fact, it is completely harmless to forget the axioms of triangulated categories and only treat them as the homotopy category of some stable ∞-categories! Although not properly justified in a chronological order, the triangulated category resembles the stable ∞-category in a usual homotopy category, where most of homotopical information is lost.

If $\mathcal{C}$ is a stable ∞-category, then the way of putting a t-structure on its homotopy category is closely related to certain localizations on $\mathcal{C}$.

Definition 6.2.17 Let $\mathcal{C}$ be a stable ∞-category, we say $\mathcal{C}$ is equipped with a t-structure if $\mathrm{h}\mathcal{C}$ is equipped with such one. Accordingly we let $\mathcal{C}_{\leq n}$ and $\mathcal{C}_{\geq n}$ be those full subcategories of $\mathcal{C}$ spanned by the objects in $(\mathrm{h}\mathcal{C})_{\leq n}$ and $(\mathrm{h}\mathcal{C})_{\geq n}$.

From the definition we have immediately the following proposition:

Proposition 6.2.18 *Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure. For every $n \in \mathbb{Z}$, $\mathcal{C}_{\leq n}$ is a localization of $\mathcal{C}$.*

Corollary 6.2.19 *Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure. The full subcategories $\mathcal{C}_{\leq n}$ are stable under all limits which exist in $\mathcal{C}$. Dually the subcategories $\mathcal{C}_{\geq n}$ are stable under all colimits which exist in $\mathcal{C}$.*

However, not every localization of $\mathcal{C}$ is related to a t-structure.

Definition 6.2.20 Let $\mathcal{C}$ be an ∞-category admits pushouts. A collection of morphisms S is quasisaturated if the following conditions are satisfied:

1. Every equivalence in $\mathcal{C}$ belongs to S.
2. Given a 2-simplex $\Delta^2 \to \mathcal{C}$, if any two of faces belong to S, so does the third.
3. Given a pushout

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X' \arrow[d, "f'"'] \\ Y \arrow[r] & Y' \end{tikzcd}$$

if $f \in S$, then $f' \in S$.

For each collection of morphisms S, there exists a smallest quasisaturated collection $\overline{S}$ containing S. We call it generated by S.

Example 6.2.21 Let $\mathcal{C}$ be an ∞-category admits pushouts, let $L : \mathcal{C} \to \mathcal{C}$ be a localization functor. Let S be the collection of morphisms f such that $L(f)$ is an equivalence. Then S is quasisaturated.

Definition 6.2.22 Let $\mathcal{C}$ be a stable ∞-category, a full subcategory $\mathcal{C}' \subset \mathcal{C}$ is closed under extension if for any distinguished triangle

$$X \to Y \to Z \to X[1] \tag{6.2.5}$$

such that $X, Z \in \mathcal{C}'$, then Y also belongs to $\mathcal{C}'$.

Proposition 6.2.23 *[[Lur17], Proposition 1.2.1.16] Let $\mathcal{C}$ be a stable ∞-category, let $L : \mathcal{C} \to \mathcal{C}$ be a localization functor. Set S as in Example 6.2.21. The followings are equivalent:*

1. *There exists a collection of morphisms $\{f : 0 \to X\}$ generating S.*
2. *The collection $\{0 \to X : L(X) \simeq 0\}$ generates S.*
3. *The essential image of L is closed under extensions.*
4. *For any $A \in \mathcal{C}, B \in L\mathcal{C}$, the natural map $\mathrm{Ext}^1(LA, B) \to \mathrm{Ext}^1(A, B)$ is injective, where $\mathrm{Ext}^1(A, B) := \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(A[-1], B)$.*

5. *The full subcategories* $\mathcal{C}_{\geq 0} = \{A : LA \simeq 0\}$ *and* $\mathcal{C}_{\leq -1} = \{A : LA \simeq A\}$ *determine a t-structure on* $\mathcal{C}$*.*

If any of these conditions is satisfied, then we call L *a t-localization.*

6.3 Examples of stable ∞-category

We still need a convincing example of stable ∞-categories, therefore, we will construct not just a single example, but a family of stable ∞-categories and a method to turn any ∞-category admitting finite (co)limits into a stable ∞-category.

Before introducing the construction, let us have a look at the functors between stable ∞-categories.

Definition 6.3.1

1. An ∞-category is said to be left exact if it admits all finite limits. A functor between left exact ∞-categories is said to be left exact if it preserves all finite limits. We denote the full subcategory of left exact functors in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ as $\mathrm{Fun}^L(\mathcal{C}, \mathcal{D})$. Dually one can define right exact categories and right exact functors.
2. A functor $F : \mathcal{C} \to \mathcal{D}$ is called exact if it is pointed (i.e., preserves zero objects) and sends exact sequence to exact sequence. We denote the full subcategory of exact functors in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ as $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})$.

Theorem 6.3.2 *[[Lur09], Corollary 4.4.2.4]*

1. *An* ∞*-category admits finite limits iff it has a terminal object and admits pullbacks.*
2. *A functor is left exact iff it preserves terminal objects and pullbacks.*

In particular, a functor between stable ∞*-categories is exact iff it is left exact iff it is right exact.*

Very roughly speaking, the stabilization of $\mathcal{C}$ is a stable ∞-category that "universally approximates" $\mathcal{C}$ from the left.

Definition 6.3.3 Let $\mathcal{C}$ be an ∞-category admits finite limits. The stabilization of $\mathcal{C}$ is a stable ∞-category $\mathrm{Stab}(\mathcal{C})$ together with a left exact functor $\Omega^\infty : \mathrm{Stab}(\mathcal{C}) \to \mathcal{C}$ such that for every stable ∞-category $\mathcal{D}$, the composition with Ω^∞ induces an equivalence

$$\mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}, \mathrm{Stab}(\mathcal{C})) \xrightarrow{\cong} \mathrm{Fun}^L(\mathcal{D}, \mathcal{C}). \tag{6.3.1}$$

We now explain how to construct such a stabilization of $\mathcal{C}$ via spectrum objects in $\mathcal{C}$. This construction matches the expectation that objects in $\mathrm{Stab}(\mathcal{C})$ should be a sequence $\{X_n\}$ such that $X_n \simeq \Omega X_{n+1}$.

Definition 6.3.4 The ∞-category of spectrum objects $\mathrm{Sp}(\mathcal{C})$ for $\mathcal{C}$ a small ∞-category is the sequential limit

$$\mathrm{Sp}(\mathcal{C}) := \lim\left(\dots \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_*\right) \tag{6.3.2}$$

in the ∞-category Cat_∞ of small ∞-categories.

This definition can be extended to any ∞-category, in particular, to the ∞-category of small animae An. We denote $\mathcal{SH} := \mathrm{Sp}(\mathrm{An})$ the ∞-category of spectra.

Proposition 6.3.5 *[[Lur17], Proposition 1.4.2.16, Proposition 1.4.2.24] The ∞-category of spectrum objects $\mathrm{Sp}(\mathcal{C})$ is stable and the universal stabilization of $\mathcal{C}$ in the sense of Definition 6.3.3.*

One can think of stable ∞-categories as an analogue of abelian groups in commutative algebra, or an analogue of chain complexes in homological algebra: It behaves much like the derived category of an abelian category. Contemplate this, we are going to justify the following definition.

Definition 6.3.6 Let $\mathcal{A}$ be an abelian category and $\mathrm{Ch}(\mathcal{A})$ the 1-category of chain complexes with values in $\mathcal{A}$. The simplicial Dold-Kan correspondence [[Lur17], Theorem 1.2.3.7] allows us to view $\mathrm{Ch}(\mathcal{A})$ as a simplicial category, and we may identify $\mathrm{Ch}(\mathcal{A})$ with its homotopy coherent nerve.
Let W_{qis} be the collection of quasi-isomorphisms in $\mathrm{Ch}(\mathcal{A})$, then the derived ∞-category of $\mathcal{A}$ is the localization with respect to this collection:

$$D(\mathcal{A}) := \mathrm{Ch}(\mathcal{A})\left[W_{\mathrm{qis}}^{-1}\right]. \tag{6.3.3}$$

Proposition 6.3.7 *[[Lur17], Proposition 1.3.5.9, Proposition 1.3.5.13] Let $\mathcal{A}$ be an abelian category, then the derived ∞-category $D(\mathcal{A})$ is presentably stable.*

6.4 Symmetric monoidal structure

We will briefly introduce two special cases of $\mathcal{E}_n$-algebras: $n = 1$ and $n = \infty$ cases. For a full treatment of the theory, we refer readers to [[Lur17], §3, §4].
We give the proper definition of fibrations of simplicial sets first. The theory of fibrations of simplicial sets was studied in [[Lur09], §2] in detail.

Definition 6.4.1 A morphism $f : X \to S$ of simplicial sets is

1. a left fibration if f has the right lifting property with respect to all horn inclusions $\Lambda^n_i \subset \Delta^n, 0 \leq i < n$.
2. a right fibration if f has the right lifting property with respect to all horn inclusions $\Lambda^n_i \subset \Delta^n, 0 < i \leq n$.
3. a Kan fibration if f is a left fibration and a right fibration.

4. an inner fibration if f has the right lifting property with respect to all horn inclusions $\Lambda_i^n \subset \Delta^n, 0 < i < n$.

Definition 6.4.2 Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories.

1. An edge $f : X \to Y$ in $\mathcal{C}$ is called F-coCartesian if the natural map $\mathcal{C}_{f\,/} \to \mathcal{C}_{X\,/} \times_{\mathcal{D}_{F(X)\,/}} \mathcal{D}_{F(f)\,/}$ is a trivial Kan fibration, i.e., a Kan fibration that is at the same time weak homotopy equivalence.
2. The functor F is called a coCartesian fibration if it is an inner fibration and for every edge $g : X \to Y$ in $\mathcal{D}$ and $X' \in \mathcal{C}$ with $F(X') = X$, there exists a F-coCartesian lift f of g with source X'.

With the notation of coCartesian fibrations, we are able to define the notion of (commutative) algebra objects in a (symmetric) monoidal ∞-category. We will use the fact that Cat_∞ admits finite products.

Recall a morphism $[n] \to [m]$ in Δ is called convex if it is injective with image consisting of consecutive integers.

Definition 6.4.3

1. A monoidal ∞-category is an ∞-category $\mathcal{C}^\otimes$ together with a coCartesian fibration $p : \mathcal{C}^\otimes \to \Delta^{\mathrm{op}}$ such that the Segal condition is fulfilled: for every $[n] \in \Delta$ the Segal map
$$(e_i^*)_{i=1}^n : \mathcal{C}^\otimes_{[n]} \to \prod_{i=1}^n \mathcal{C}^\otimes_{[1]} \tag{6.4.1}$$
induced by the n inclusion maps $e_i : [1] \cong \{i-1 \le i\} \hookrightarrow [n]$ is an equivalence. Here we denote $\mathcal{C}^\otimes_{[n]}$ to be the fiber of p over $[n]$, viewed as a subcategory of $\mathcal{C}^\otimes$. In such case, we say that p induces a monoidal structure on $\mathcal{C} := \mathcal{C}^\otimes_{[1]}$.
2. A monoidal functor between monoidal ∞-categories $p : \mathcal{C}^\otimes \to \Delta^{\mathrm{op}}$ and $q : \mathcal{D}^\boxtimes \to \Delta^{\mathrm{op}}$ is a functor sending p-coCartesian edges to q-coCartesian edges. More generally, a lax monoidal functor is a functor sending p-coCartesian lifts of convex morphisms to q-coCartesian edges.
3. An $\mathcal{E}_1$-algebra in $\mathcal{C}^\otimes$ a monoidal category is a lax monoidal functor $\Delta^{\mathrm{op}} \to \mathcal{C}^\otimes$, i.e., a simplicial object in $\mathcal{C}^\otimes$ such that the Segal condition is fulfilled. We denote all $\mathcal{E}_1$-algebras in $\mathcal{C}^\otimes$ to be $\mathrm{Alg}(\mathcal{C}^\otimes)$.

The definition of a symmetric monoidal ∞-category is similarly to the monoidal one, but we need to use another index category that encodes the commutative property.

Let Fin_* be the (nerve of) category of all pointed finite sets and set-theoretic maps. We write $\{n\} := \{*, 1, ..., n\}$ and call a morphism $\{n\} \to \{m\}$ inert if the preimage of elements that are different from the base point is a singleton.

Definition 6.4.4

1. A symmetric monoidal ∞-category is an ∞-category $\mathcal{C}^{\otimes}$ together with a coCartesian fibration $p : \mathcal{C} \to \mathrm{Fin}_*$ such that the Segal condition is fulfilled: for every $\{n\} \in \mathrm{Fin}_*$ the Segal map
$$(\rho_i^*)_{i=1}^n : \mathcal{C}^{\otimes}_{\{n\}} \to \prod_{i=1}^{n} \mathcal{C}^{\otimes}_{\{1\}} \tag{6.4.2}$$
induced by the n fold maps $\rho_i : \{n\} \to \{1\}$, which sends i to 1 and rest to $*$, is an equivalence. In such case, we say that p induces a symmetric monoidal structure on $\mathcal{C} := \mathcal{C}^{\otimes}_{\{1\}}$.
2. A symmetric monoidal functor between symmetric monoidal ∞-categories $p : \mathcal{C}^{\otimes} \to \mathrm{Fin}_*$ and $q : \mathcal{D}^{\boxtimes} \to \mathrm{Fin}_*$ is a functor sending p-coCartesian edges to q-coCartesian edges. More generally, a lax symmetric monoidal functor is a functor sending p-coCartesian lifts of inert morphisms to q-coCartesian edges.
3. An $\mathcal{E}_\infty$-algebra, or commutative algebra in $\mathcal{C}^{\otimes}$ a symmetric monoidal category is a lax symmetric monoidal functor $\mathrm{Fin}_* \to \mathcal{C}^{\otimes}$. We denote all $\mathcal{E}_\infty$-algebras in $\mathcal{C}^{\otimes}$ to be $\mathrm{CAlg}(\mathcal{C}^{\otimes})$.

There is a certain kind of symmetric monoidal structures that is important for us.

Proposition 6.4.5 *[[Lur17], Proposition 2.4.1.5] If $\mathcal{C}$ is an ∞-category with finite products, then there is a coCartesian fibration $\mathcal{C}^{\times} \to \mathrm{Fin}_*$ making $\mathcal{C}^{\times}$ a symmetric monoidal ∞-category. We call this symmetric monoidal structure the Cartesian monoidal structure on $\mathcal{C}$.*

Proposition 6.4.6 *Suppose $\mathcal{C}^{\times}, \mathcal{D}^{\times}$ are equipped with the Cartesian monoidal structure, then a functor $F : \mathcal{C} \to \mathcal{D}$ is symmetric monoidal if and only if F preserves finite products.*

If we equip the ∞-category with Cartesian monoidal structure, then the commutative algebra objects in it are also called commutative monoids.

Example 6.4.7 [[Lur17], Example 2.2.6.9] Let $\mathcal{C}^{\otimes}, \mathcal{D}^{\boxtimes}$ be symmetric monoidal ∞-categories. Suppose $\mathcal{D}^{\boxtimes}$ admits all small colimits and the symmetric monoidal structure on $\mathcal{D}$ preserves them in each variable. Then there is a symmetric monoidal structure on $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, called the Day convolution. Moreover, the commutative algebra objects in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ are precisely those lax symmetric monoidal functors.

6.5 Presentable ∞-category

We begin by defining the ∞-category of presheaves and state an ∞-categorical Yoneda lemma. As we have seen in Proposition 6.1.17, the Yoneda embedding should take values in the ∞-category of presheaves of spaces, which work as a generalized notion of sets.

Definition 6.5.1 For $\mathcal{C}$ an ∞-category, the category of presheaves of $\mathcal{C}$ is $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$, we denote it by $\mathcal{P}(\mathcal{C})$.

Under this definition, Lurie proves the for any $\mathcal{C}$ an ∞-category, $\mathcal{P}(\mathcal{C})$ admits small limits and colimits [[Lur09], Corollary 5.1.2.4].
The construction of a Yoneda embedding is cumbersome [[Lur09], §5.1.3], but we have can still get a nice result as in 1-category.

Proposition 6.5.2 *[[Lur09], Proposition 5.1.3.1] Let $\mathcal{C}$ be an ∞-category. The Yoneda embedding*

$$よ : \mathcal{C} \to \mathcal{P}(\mathcal{C}) \tag{6.5.1}$$

is fully faithful.

Finally we describe a universal property of the presheaf category.

Theorem 6.5.3 *[[Lur09], Theorem 5.1.5.6] Let $\mathcal{C}$ be a small ∞-category and $\mathcal{D}$ be an ∞-category admit small colimits, then the Yoneda embedding $よ : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ induces an equivalence of ∞-categories:*

$$\mathrm{Fun}^L(\mathcal{P}(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \tag{6.5.2}$$

where Fun^L is the category of all colimit preserving functors.

A close related concept with presheaves is the so called presentable ∞-categories. This relies on the fact that even sometimes $\mathcal{P}(\mathcal{C})$ is not small, we can look at some subcategories that can be controlled with small compact objects with colimits.

Definition 6.5.4 Let κ be a regular cardinal. An ∞-category $\mathcal{C}$ is called κ-accessible if $\mathcal{C}$ is locally small, has all κ-filtered colimits and $\mathcal{C}$ is generated under κ-filtered colimits of κ-compact objects, which form a essentially small subcategory of $\mathcal{C}$.

Definition 6.5.5 An ∞-category $\mathcal{C}$ is called presentable if it is accessible and admits all small colimits.

We note that for any $\mathcal{C}$ an ∞-category, $\mathcal{P}(\mathcal{C})$ is always presentable. In general however, there is a nice characterization of all presentable ∞-categories as the accessible localizations of some presheaves.

Lemma 6.5.6 *[[Lur09], Lemma 5.5.1.4] Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories with filtered colimits. Let G be a right adjoint of F. Then G preserves filtered colimits if and only if F preserves compact objects.*

Theorem 6.5.7 *[[Lur09], Theorem 5.5.1.1] Let $\mathcal{C}$ be an ∞-category, then the followings are equivalent:*

1. *$\mathcal{C}$ is presentable.*
2. *There exists a small ∞-category $\mathcal{D}$ such that $\mathcal{C}$ is an accessible localization of $\mathcal{P}(\mathcal{D})$.*

Now we state a very useful statement about the existence of adjoints, the adjoint functor theorem:

Theorem 6.5.8 *[[Lur09], Corollary 5.5.2.9] Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between presentable ∞-categories.*

1. *F has a right adjoint if and only if F preserves colimits.*
2. *F has a left adjoint if and only if F preserves limits.*

The following three propositions are about presentable stable ∞-categories:

Proposition 6.5.9 *[[Lur17], Proposition 1.4.4.4] Let $\mathcal{C}$ be a presentable ∞-category, then the functor $\Omega^\infty : \mathrm{Stab}(\mathcal{C}) \to \mathcal{C}_*$ admits a left adjoint $\Sigma^\infty : \mathcal{C}_* \to \mathrm{Stab}(\mathcal{C})$.*

Theorem 6.5.10 *[[Lur17], Proposition 1.4.4.9] Let $\mathcal{C}$ be an ∞-category, then the followings are equivalent:*

1. *$\mathcal{C}$ is stable and presentable.*
2. *There exists a presentable stable ∞-category $\mathcal{D}$ such that $\mathcal{C}$ is an accessible left-exact localization of $\mathcal{D}$.*

Proposition 6.5.11 *[[Lur17], Proposition 1.4.4.11] Let $\mathcal{C}$ be a presentable stable ∞-category. Let $\{X_\alpha\}$ be a small collection of objects in $\mathcal{C}$. The smallest full subcategory of $\mathcal{C}$ generated by X_α under small colimits and extensions is presentable.*

The next two lemmata are about sifted colimits and spherical presheaves, which are used in the construction of Thom spectra.

Definition 6.5.12

1. A non-empty ∞-category $\mathcal{C}$ is sifted if the diagonal functor $\Delta : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ is a cofinal functor, i.e., for any functor $F : \mathcal{C} \times \mathcal{C} \to \mathcal{D}$, pre-composing with Δ preserves colimit:
$$\mathrm{colim}\Big(\mathcal{C} \xrightarrow{\Delta} \mathcal{C} \times \mathcal{C} \xrightarrow{F} \mathcal{D}\Big) \simeq \mathrm{colim}\Big(\mathcal{C} \times \mathcal{C} \xrightarrow{F} \mathcal{D}\Big) \tag{6.5.3}$$
when either of the limits exists.
2. A sifted colimit in an ∞-category $\mathcal{C}$ is the colimit of diagrams $F : \mathcal{D} \to \mathcal{C}$, where $\mathcal{D}$ is a sifted ∞-category. All ∞-categories admitting sifted colimits span a subcategory $\mathrm{Cat}_\infty^{\mathrm{sift}}$ in Cat_∞.

Proposition 6.5.13 *[[Lur09], Prop. 5.5.8.15] The inclusion $\mathrm{Cat}_\infty^{\mathrm{sift}} \to \mathrm{Cat}_\infty$ has a left adjoint $\mathcal{P}_\Sigma : \mathrm{Cat}_\infty \to \mathrm{Cat}_\infty^{\mathrm{sift}}$, which associates an ∞-category to the presheaves $\mathcal{C}^{\mathrm{op}} \to \mathrm{An}$ that sends finite coproducts to products, called the ∞-category of spherical presheaves.*

Definition 6.5.14 An ∞-topos $\mathcal{X}$ is an accessible left exact localization of $\mathcal{P}(\mathcal{C})$ where $\mathcal{C}$ is a small ∞-category.

6.6 Category of spans

Let $\mathcal{C}$ be an ∞-category and $\mathrm{left}, \mathrm{right}$ be two collections of morphisms in $\mathcal{C}$, such that they contain equivalences, are closed under composition and closed under pull back along each other.

Definition 6.6.1 The ∞-category $\mathrm{Span}(\mathcal{C}, \mathrm{left}, \mathrm{right})$ is the category with objects the same as $\mathcal{C}$, and morphisms from X to Z are spans like $X \overset{f}{\leftarrow} Y \overset{g}{\to} Z$ such that f is a left morphism and g is a right morphism. The composition of $X \overset{f}{\leftarrow} Y \overset{g}{\to} Z$ and $Z \overset{h}{\leftarrow} W \overset{j}{\to} A$ is given by

$$\begin{tikzcd}
Y \times_Z W \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & W \arrow[r, "j"] \arrow[d, "h"'] & A \\
Y \arrow[r, "g"] \arrow[d, "f"'] & Z & \\
X & &
\end{tikzcd}$$

Definition 6.6.2 An ∞-category $\mathcal{C}$ is called extensive, if $\mathcal{C}$ admits finite coproducts and they are disjoint (i.e., for every $X, Y \in \mathcal{C}$, $X \times_{X \sqcup Y} Y$ is an initial object), and their decompositions are stable under pullbacks.

Notation 6.6.3 We will mainly consider $\mathcal{C} = \mathcal{S}\mathrm{m}_S$ the category of smooth schemes over S (note $\mathcal{S}\mathrm{m}_S$ is extensive) and following collections of morphisms:

1. all = all morphisms in $\mathcal{C}$,
2. inj = all injective maps in $\mathcal{C}$,
3. Suppose $\mathcal{C}$ is extensive, then we denote fold to be the classes of maps that are finite sums of fold maps $X^{\coprod n} \to X$.
4. For simplicity we write $\mathrm{Span}(\mathcal{C}) = \mathrm{Span}(\mathcal{C}, \mathrm{all}, \mathrm{all})$.

Let $\mathcal{D}$ be an ∞-category with finite products, then functors $\mathrm{Span}(\mathrm{Fin}) \to \mathcal{D}$ preserving finite products are precisely the commutative monoids in $\mathcal{D}$.

Proposition 6.6.4 *[[BH21], Proposition C.1]*

1. *There is an equivalence of ∞-categories*

$$\begin{gathered}\mathrm{Fin}_* \simeq \mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all}) \\ X_+ \mapsto X, (f : X_+ \to Y_+) \mapsto \left(X \hookleftarrow f^{-1}(Y) \xrightarrow{f} Y\right).\end{gathered} \tag{6.6.1}$$

2. *Let $\mathcal{D}$ be an ∞-category with finite products, then the restriction of*

$$\mathrm{Fun}(\mathrm{Span}(\mathrm{Fin}), \mathcal{D}) \to \mathrm{Fun}(\mathrm{Fin}_*, \mathcal{D}) \tag{6.6.2}$$

onto

$$\mathrm{Fun}^\times(\mathrm{Span}(\mathrm{Fin}), \mathcal{D}) \xrightarrow{\simeq} \mathrm{CAlg}(\mathcal{D}) \tag{6.6.3}$$

is an equivalence of ∞-categories.

Now if $\mathcal{C}$ is extensive and $\mathcal{D}$ admits finite products, we have an obvious functor

$$\Theta : \mathcal{C}^{\mathrm{op}} \times \mathrm{Span}(\mathrm{Fin}) \to \mathrm{Span}(\mathcal{C}, \mathrm{all}, \mathrm{fold}), (X, F) \mapsto \coprod_F X. \tag{6.6.4}$$

We can generalize the above proposition to spherical presheaves using Θ.

Proposition 6.6.5 *[[BH21], Proposition C.5] Let $\mathcal{C}$ be an extensive ∞-category and $\mathcal{D}$ an ∞-category with finite products. The functor*

$$\Theta^* : \mathrm{Fun}(\mathrm{Span}(\mathcal{C}, \mathrm{all}, \mathrm{fold}), \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathrm{Span}(\mathrm{Fin}), \mathcal{D}) \tag{6.6.5}$$

restricts to an equivalence of ∞-categories

$$\mathrm{Fun}^\times(\mathrm{Span}(\mathcal{C}, \mathrm{all}, \mathrm{fold}), \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}^\times(\mathcal{C}^{\mathrm{op}}, \mathrm{CAlg}(\mathcal{D})). \tag{6.6.6}$$

We need a relation of spans with Kan extension. For a functor $F : \mathrm{Span}(\mathcal{C}, \mathrm{left}, \mathrm{right}) \to \mathcal{D}$, we let $F|\mathcal{C} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ be the restriction onto left morphisms.

Proposition 6.6.6 *Let $\mathcal{C}, \mathcal{D}$ be ∞-categories and m be a collection of morphisms in $\mathcal{C}$ that is closed under pull back and composition. Let $\mathcal{C}_0$ a full subcategory of $\mathcal{C}$ such that if $X \in \mathcal{C}_0$ and $Y \to X$ is a m-morphism, then $Y \in \mathcal{C}_0$. A functor $F : \mathrm{Span}(\mathcal{C}_0, \mathrm{all}, m) \to \mathcal{D}$ has a right Kan extension to $\mathrm{Span}(\mathcal{C}, \mathrm{all}, m) \to \mathcal{D}$ iff the functor $F|\mathcal{C}_0 : \mathcal{C}_0^{\mathrm{op}} \to \mathcal{D}$ has a right Kan extension to $\mathcal{C}$.*

Proof. For $X \in \mathcal{C}$, the inclusion

$$\mathcal{C}_0 \times_{\mathcal{C}} \mathcal{C}_{X\,/} \hookrightarrow \mathrm{Span}(\mathcal{C}_0, \mathrm{all}, m) \times_{\mathrm{Span}(\mathcal{C},\ \mathrm{all}, m)} \mathrm{Span}(\mathcal{C}, \mathrm{all}, m)_{X\,/} \tag{6.6.7}$$

has a right adjoint and hence is coinitial, This proves the statement. □

6.7 Pro-objects

We introduce pro objects, which is a generalization of presheaves.

Proposition 6.7.1 *[[Lur09], Proposition 5.3.6.2] Let $\mathcal{C}$ be an ∞-category, there is an ∞-category $\mathrm{Pro}(\mathcal{C})$ and a embedding $j : \mathcal{C} \to \mathrm{Pro}(\mathcal{C})$ with following universal properties:*

1. *$\mathrm{Pro}(\mathcal{C})$ has all small cofiltered limits.*
2. *Let $\mathcal{D}$ be an ∞-category with small cofiltered limits, let $\mathrm{Fun}'(\mathcal{C}, \mathcal{D})$ be those functors that preserve small cofiltered limits, then the embedding j induces an equivalence*

$$\mathrm{Fun}'(\mathrm{Pro}(\mathcal{C}), \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \tag{6.7.1}$$

If $\mathcal{C}$ is accessible, we may identify $\mathrm{Pro}(\mathcal{C})$ with the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathrm{An})^{\mathrm{op}}$ spanned by functors that are left-exact and accessible.

Proposition 6.7.2 *[[Lur09], Proposition 5.3.1.16] Every pro-object $X \in \mathrm{Pro}(\mathcal{C})$ can be corepresented by a diagram $\mathcal{J} \to \mathcal{C}$ where $\mathcal{J}$ is a small cofiltered partially ordered set.*

We will refer to Pro(An) as the ∞-category of shapes, this name will be justified right now.
Let $G : \mathcal{D} \to \mathcal{C}$ be an ∞-functor of presentable ∞-categories preserving finite limits, a "slight variant" of adjoint functor theorem gives us a pro-left adjoint $F : \mathcal{C} \to \mathrm{Pro}(\mathcal{D})$ to G. This applies to the setting of ∞-topoi. The following definition is a special case of shape theory as in [[Lur09], §7.1.6].

Definition 6.7.3 Let $\mathcal{X}$ be an ∞-topos, let $\pi : \mathcal{X} \to \mathrm{An}$ be the unique geometric morphism, π_* the direct image functor. Let π^* be the left adjoint to π_*. The above discussion ensures the existence of a pro-left-adjoint $\pi_! : \mathcal{X} \to \mathrm{Pro}(\mathrm{An})$.
The fundamental pro-∞-groupoid of $\mathcal{X}$ is the shape

$$\Pi_\infty \mathcal{X} := \pi_! \mathbb{1} \in \mathrm{Pro}(\mathrm{An}) \tag{6.7.2}$$

In other words, it is the composition $\pi_* \pi^* : \mathrm{An} \to \mathrm{An}$.

6.8 $\mathcal{E}_\infty$-descendability

On the level of $\mathcal{E}_\infty$-rings, we have a notion of descendable objects, inspired by the concept in obstruction theory. This is firstly introduced in [Mat16] and get a slight generalization in [AS25].

Definition 6.8.1 Let $f : R \to S$ be a morphism of $\mathcal{E}_\infty$-rings. f is said to be $\mathcal{E}_\infty$-descendable if the map of towers $\{R\} \to \{\mathrm{Tot}^n(S^{\wedge *+1})\}_n$ is a pro-equivalence in $\mathrm{Pro}(\mathrm{CAlg}(\mathrm{Mod}_R))$.

We only need to use an equivalent characterization of $\mathcal{E}_\infty$-descendability for our purpose.

Proposition 6.8.2 *[[AS25], Prop. 2.3] Let $f : R \to S$ be a morphism of $\mathcal{E}_\infty$-rings. The followings are equivalent:*

1. *f is $\mathcal{E}_\infty$-descendable.*
2. *The map $R \to \mathrm{Tot}^n(S^{\wedge *+1})$ admits an $\mathcal{E}_\infty$-retraction for some $n \geq 0$.*
3. *If $\mathcal{C}$ is the smallest full subcategory of $\mathrm{CAlg}(\mathrm{Mod}_R)$ which contains the $\mathcal{E}_\infty$-algebra that admits a map from S and $\mathcal{C}$ is closed under finite limits and retractions, then $\mathcal{C}$ contains R.*

Proposition 6.8.3 *[[AS25], Lemma 2.5] Let $F : \mathcal{C} \to \mathcal{D}$ be an exact lax symmetric monoidal functor of symmetric monoidal stable ∞-categories. If $f : R \to S$ is $\mathcal{E}_\infty$-descendable, then so is $F(f) : F(R) \to F(S)$.*

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