UNIVERSITÄT HEIDELBERG
ZUKUNFT SEIT 1386

Slice, décale and realize, motivically

Cheni Yuki Yang

Universität Heidelberg

2026-01-29

1. Motivation

1.1 Adams-Novikov spectral sequences

Classical Adams spectral sequence takes the ordinary homology mod p as an input and approximates the p-stem of the stable homotopy groups of spheres.

We can generalize this to any ring spectrum E, inducing E-Adams spectral sequences. Suppose we want to approximate $[\mathbb{S}, \mathbb{S}]$, we may try to make use of $E_*(\mathbb{S}) \to E_*(\mathbb{S})$ over $E_*(E)$ the Hopf algebroid of dual E-Steenrod operations.

If we view it as a base change $\mathbb{S} \to E$, then we can recover the module information over $\mathbb{S}$ by totalization of the cobar resolution

$$\mathbb{S} \to E \rightrightarrows E \wedge E \substack{\rightarrow\\\rightarrow\\\rightarrow} \dots$$

This is a cosimplicial spectrum in $\mathrm{Mod}_{\mathbb{S}}$, and carries a filtration by finite totalizations. The E-Adams spectral sequence is the spectral sequence associated to this filtration.

1.1 Adams-Novikov spectral sequences

Specialize to $E = \mathrm{MU}$, the complex cobordism spectrum, we get a spectral sequence which converges faster to the full stable homotopy group of spheres. We call this the Adams-Novikov spectral sequence, which plays an important role in chromatic homotopy theory.

$$E_1^{s,t}(AN) := \pi_{-s-t}\mathrm{gr}^s\mathrm{Tot}_\star(\mathrm{MU}^{\wedge *+1}) \Rightarrow \pi_{-s-t}(\mathbb{S})$$

We want to extend this result, by considering a cobordism theory of “motivic nature” and defining filtration on it. We hope it also converges to certain homotopy groups of connective spectra, or best, of motivic spheres.

1.2 Main result

The stable homotopy group of motivic spheres has its own interest due to rich algebraic and arithmetic information contained in it, e.g. the Milnor-Witt K-theory. Inspired by topological setting, we can compare it with stable homotopy group of spheres in the form of spectral sequences.

Theorem 1.2.1 The Adams-Novikov spectral sequence $E_r^{s,t}(AN)$ and the motivic Atiyah-Hirzebruch spectral sequence

$$E_1^{p,q}(AH) = \pi_{-p-q,0}(\mathrm{gr}^{-q}\mathbb{S}_k^{\star})(k) \Longrightarrow \pi_{-p-q,0}(\mathbb{S}_k)(k)$$

are isomorphic by

$$\oplus_{p,q}\, \gamma_r^{p,q} : \left(\oplus_{p,q}\, E_r^{p,q}(AH), d_r\right) \to \left(\oplus_{p,q}\, E_{2r+1}^{3p+q,2p}(AN), d_{2r+1}\right).$$

1.3 Remark on ∞-category

Axiom 1.3.1 There are model independent objects called ∞-categories with following features:

1. It has objects and morphisms between objects.
2. It contains n-isomorphisms expressing the equality of $(n-1)$-isomorphisms, where 1-isomorphims are those between objects.
3. n-isomorphisms can compositite, forming a generalization of commutative triangles and commutative squares.
4. Every 1-category gives rise to an ∞-category.
5. There are ∞-categories Cat_∞ and An, called the ∞-category of small ∞-categories and small animae, where an anima can be understood as the homotopy type of a space.
6. Cat_∞ has $\emptyset$ as initial object and $[0] = \{*\}$ as terminal object. It has products, pullbacks and pushouts.

1.3 Remark on ∞-category

7. There're categorical constructions like subcategory and Dwyer-Kan localization with respect to certain collection of morphisms.

2. Motivic homotopy theory

2.1 Motivic space

Motivic homotopy theory is the homotopy theory of smooth schemes where $\mathbb{A}^1$ is the interval object. In order to get an analog of smooth manifolds, we consider the presheaves of animae on $\mathcal{S}\mathrm{m}_k$ and localize them. We use Nisnevich topology because it gives us a descent of algebraic K-theory (s. Filtration).

Definition 2.1.1 The unstable motivic category $\mathcal{H}(k)$ is the localization of $\mathcal{P}(\mathcal{S}\mathrm{m}_k)$ under the following morphisms:

1. ($\mathbb{A}^1$-invariance) $X \times \mathbb{A}^1_k \to X$;
2. $よ(U) \coprod_{よ(U \times_X V)} よ(V) \to よ(X)$ for any Nisnevich square $\{U \to X, V \to X\}$;
3. the unique map $\emptyset \to よ(\emptyset)$.

where $よ : \mathcal{S}\mathrm{m}_S \to \mathcal{P}(\mathcal{S}\mathrm{m}_k)$ is the Yoneda embedding functor. This gives us a localization functor $L_{\mathrm{mot}} : \mathcal{P}(\mathcal{S}\mathrm{m}_k) \to \mathcal{H}(k)$.

2.1 Motivic space

Proposition 2.1.2 There is a closed symmetric monoidal structure on $\mathcal{H}(k)_*$ given by the localization of the section-wise smash product, where $\mathcal{H}(k)_*$ is the category of pointed motivic spaces.

2.2 ∞-category and stabilization

Just like motivic spaces resemble the classical homotopy category of spaces, the stable motivic category is a generalization of topological spectra.

In the setting of higher algebra, there is a canonical way to stabilize an ∞-category, by considering the spectrum objects $\mathrm{Sp}(\mathcal{C})$ of the category $\mathcal{C}$.

Definition 2.2.1 The ∞-category of spectrum objects $\mathrm{Sp}(\mathcal{C})$ for $\mathcal{C}$ a small ∞-category is the sequential limit

$$\mathrm{Sp}(\mathcal{C}) := \lim\left(\ldots \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_*\right)$$

in the ∞-category Cat_∞ of small ∞-categories.

2.2 ∞-category and stabilization

We have two options: invert $\mathbb{S}^1 = \Sigma(* \coprod *)$ or invert $\mathbb{P}^1 \simeq \mathbb{S}^1 \wedge \mathbb{G}_m$. The later turns out to be a better candidate.

Definition 2.2.2 The stable motivic category $\mathcal{SH}(k)$ is the colimit of the following sequence:

$$\mathcal{H}(k)_* \overset{\mathbb{P}^1 \wedge -}{\to} \mathcal{H}(k)_* \overset{\mathbb{P}^1 \wedge -}{\to} \mathcal{H}(k)_* \overset{\mathbb{P}^1 \wedge -}{\to} \dots$$

together with a symmetric monoidal functor $\Sigma^\infty_{\mathbb{P}^1} : \mathcal{H}(k)_* \to \mathcal{SH}(k)$ which sends $\mathbb{P}^1$ to an invertible object. Moreover, $\mathcal{SH}(k)$ carries a canonical symmetric monoidal structure.

We have the following universal property of $\mathcal{SH}(k)$.

2.2 ∞-category and stabilization

Proposition 2.2.3 (Robalo) The stable motivic category $\mathcal{SH}(k)$ is indeed stable. Moreover, let $\mathcal{C}$ be a pointed presentable symmetric monoidal ∞-category, the composition with stabilization

$$\mathrm{Fun}^{\otimes, L}(\mathcal{SH}(k), \mathcal{C}) \to \mathrm{Fun}^{\otimes}(\mathcal{H}(k)_*, \mathcal{C}), F \mapsto F \circ \Sigma^{\infty}_{\mathbb{P}^1}$$

is fully faithful with essential image consisting of those symmetric monoidal functors $F : \mathcal{H}(k)_* \to \mathcal{C}$ which are $\mathbb{P}^1$-stable, i.e. the homotopy cofiber of $F(\mathrm{Spec}\ k) \to F(\mathbb{P}^1)$ induced from $\mathrm{Spec}\ k \overset{\infty}{\to} \mathbb{P}^1$ is $\otimes$-invertible.

2.3 Homotopy sheaf and connectivity

Definition 2.3.1 Let $E \in \mathcal{SH}(k)$, we shall denote $\mathbb{S}^{i,j} := (\mathbb{S}^1)^{\wedge i-j} \wedge \mathbb{G}_m^{\wedge j}$ to be the motivic (i,j)-sphere. The (i,j)-th homotopy sheaf $\pi_{i,j}(E)$ of E is the sheafification of the presheaf

$$X \in \mathcal{S}\mathrm{m}_k \mapsto [\Sigma^\infty_{\mathbb{P}^1} X_+, E \wedge \mathbb{S}^{-i,-j}]_{\mathcal{SH}(k)}$$

where $[X, Y]$ is the 0-th truncation of the mapping space, i.e. the set of morphisms in the homotopy category.

We may discuss the connectivity with respect to this bigraded homotopy sheaf.

2.3 Homotopy sheaf and connectivity

Definition 2.3.2 The homotopy t-structure on $\mathcal{SH}(k)$ is given by

$$\mathcal{SH}(k)_{\geq 0} = \left\{E \in \mathcal{SH}(k) : \pi_{i,j}(E) = 0, \forall i - j < 0\right\}$$

$$\mathcal{SH}(k)_{\leq 0} = \left\{E \in \mathcal{SH}(k) : \pi_{i,j}(E) = 0, \forall i - j > 0\right\}.$$

Theorem 2.3.3 The homotopy t-structure is indeed a t-structure. And all truncation functors are symmetric monoidal with respect to the smash product on $\mathcal{SH}(k)$.

3. Algebraic cobordism

3.1 Vector bundles and Thom spaces

For a vector bundle $\mathcal{E} \to X$ over smooth scheme $X \in \mathcal{S}\mathrm{m}_k$, we can define the Thom space $\mathrm{Th}(\mathcal{E}) := \mathcal{E}/(\mathcal{E} - X) \in \mathrm{Shv}_{\mathrm{Nis}}(\mathcal{S}\mathrm{m}_k)_*$ pointed at the image of $\mathcal{E} - X$, where X is embedded as the zero section. Suppose we have $\mathcal{E}_1 \to X_1$ and $\mathcal{E}_2 \to X_2$, then

$$\mathrm{Th}(\mathcal{E}_1 \times \mathcal{E}_2 \to X_1 \times X_2) = \mathrm{Th}(\mathcal{E}_1) \wedge \mathrm{Th}(\mathcal{E}_2).$$

Definition 3.1.1 Let

$$\mathrm{BGL}_n = \mathrm{Gr}_n := \lim_k \mathrm{Gr}_n\left(\mathbb{A}^{n+k}\right)$$

be the Grassmannian of n-dimensional affine subspaces and γ_n the tautological bundle on it. The product $\mathbb{A}^1 \times \gamma_n \to \mathrm{BGL}_n$ is classified by pullback of the canonical map $\mathrm{BGL}_n \to$

3.1 Vector bundles and Thom spaces

BGL_{n+1} and γ_{n+1}, this induces a bundle map $\mathbb{A}^1 \times \gamma_n \to \gamma_{n+1}$, and a structure map

$$\mathrm{Th}(\mathbb{A}^1) \wedge \mathrm{Th}(\gamma_n) \simeq \mathbb{P}^1 \wedge \mathrm{Th}(\gamma_n) \to \mathrm{Th}(\gamma_{n+1}).$$

The algebraic cobordism spectrum $\mathrm{MGL}_k \in \mathcal{SH}(k)$ is defined to be

$$\mathrm{MGL}_k := \mathrm{colim}_n \Sigma_{\mathbb{P}^1}^{-n} \Sigma_{\mathbb{P}^1}^{\infty} \mathrm{Th}(\gamma_n).$$

Definition 3.1.2 the algebraic K-theory spectrum $\mathrm{KGL} \in \mathcal{SH}(k)$ as

$$\mathrm{KGL} := \Sigma_{\mathbb{P}^1}^{\infty} L_{\mathrm{mot}}(\mathbb{Z} \times \mathrm{BGL})$$

3.1 Vector bundles and Thom spaces

where BGL is the sequential limit of

$$\ldots \hookrightarrow \mathrm{BGL}_n \hookrightarrow \mathrm{BGL}_{n+1} \hookrightarrow \ldots$$

and the structure map is given by

$$\beta : \mathbb{P}^1 \wedge L_{\mathrm{mot}}(\mathbb{Z} \times \mathrm{BGL}) \to L_{\mathrm{mot}}(\mathbb{Z} \times \mathrm{BGL})$$

representing the Bott element in $K^0(\mathbb{P}^1 \wedge L_{\mathrm{mot}}(\mathbb{Z} \times \mathrm{BGL}))$.

3.2 Functorial treatment

Theorem 3.2.1 (Bachmann-Hoyois) There is a colimit-preserving functor $M_k : \mathcal{P}_\Sigma\left((\mathcal{S}\mathrm{m}_k)_{/\mathcal{SH}}\right) \to \mathcal{SH}(k)$, called the motivic Thom spectrum functor, which sends a spherical presheaf into a Thom spectrum.

Let $\mathrm{Vect}(X)$ be the symmetric monoidal ∞-category of vector bundles over $X \in \mathcal{S}\mathrm{m}_k$, with symmetric monoidal structure given by direct sums. We have a symmetric monoidal functor

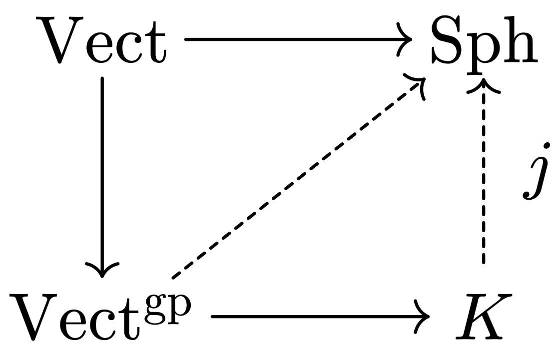
$$\mathrm{Vect}(X) \to \mathrm{Sph}(X), \xi \mapsto \Sigma^\infty_{\mathbb{P}^1} \mathrm{Th}(\xi)$$

natural in X, where $\mathrm{Sph}(X)$ is the anima spanned by invertible objects in $\mathcal{SH}(X)^\simeq$. And this gives us a natural transformation

$$\mathrm{Vect} \to \mathrm{Sph} : \mathrm{Span} \to \mathrm{CAlg}(\mathrm{An}).$$

3.2 Functorial treatment

Since $\mathrm{Sph}(X)$ is an anima, by taking the group completion of $\mathrm{Vect}(X)$, we have the factorization



Definition 3.2.2 The above natural transformation $j : K \to \mathrm{Sph}$ is called the motivic J-homomorphism.

Proposition 3.2.3 Let $e : K^\circ \hookrightarrow K$ be the injection of rank 0 part of algebraic K-theory. The bundle $\gamma : \mathrm{BGL} \to K^\circ$

3.2 Functorial treatment

representing $\mathbb{A}^1$-localization of tautological bundle induces an equivalence in $\mathcal{SH}(k)$:

$$\mathrm{MGL}_k = M_k(j \circ e \circ \gamma) \simeq M_k(j \circ e).$$

Corollary 3.2.3.1 The algebraic cobordism spectrum is equipped with an $\mathcal{E}_\infty$-ring structure.

Proof. Since M_k is colimit preserving, it sends objects in $\mathrm{CAlg}\left(\mathcal{P}_\Sigma(\mathcal{S}\mathrm{m}_k)_{/\mathcal{SH}^\simeq}\right)$ to objects in $\mathrm{CAlg}(\mathcal{SH}(k))$. It remains to check that the motivic J-homomorphism on zeroth summand is a commutative algebra object in $\mathcal{P}_\Sigma(\mathcal{S}\mathrm{m}_k)_{/\mathcal{SH}^\simeq}$, which is clear. □

4. Filtrations

4.1 Slice filtrations

Definition 4.1.1

1. The full subcategory of $\mathcal{SH}(S)$ generated by spectra $\Sigma^n_{\mathbb{S}^1}\Sigma^\infty_{\mathbb{P}^1}X_+, n \in \mathbb{Z}$ under colimits, where $X \in \mathcal{S}\mathrm{m}_S$ a smooth scheme, is called the category of effective motivic spectra, denoted by $\mathcal{SH}^{\mathrm{eff}}(S)$.
2. For any $k \in \mathbb{Z}$, the category of k-effective spectra $\mathcal{SH}^{\mathrm{eff}}(S)(k)$ is generated by

$$\Sigma^n_{\mathbb{S}^1}\Sigma^\infty_{\mathbb{P}^1}\left(\left(\mathbb{P}^1\right)^{\wedge k} \wedge X\right)_+, n \in \mathbb{Z}.$$

   where $X \in \mathcal{S}\mathrm{m}_S$ a smooth scheme.

Note that $\mathcal{SH}^{\mathrm{eff}}(S)$ is stable and presentable. By the usual adjoint functor theorem we have:

4.1 Slice filtrations

Proposition 4.1.2 The inclusion functor $\iota_k : \mathcal{SH}^{\text{eff}}(S)(k) \to \mathcal{SH}(S)$ admits a right adjoint $r_k : \mathcal{SH}(S) \to \mathcal{SH}^{\text{eff}}(S)(k)$.

For a motivic spectrum E, we set $E^k := \iota_k(r_k(E))$, called the k-effective cover of E. We then have a filtration

$$\dots \to E^{k+1} \to E^k \to E^{k-1} \to \dots$$

called the slice filtration of motivic spectra. The graded piece is called slice.

The effective spectra category expresses certain connectivity with respect to $\mathbb{G}_m$, but not $\mathbb{S}^1$. We thus can also consider an variant.

4.1 Slice filtrations

Definition 4.1.3

1. The full subcategory in $\mathcal{SH}(S)$ generated under colimits by spectra $\Sigma^n_{\mathbb{S}^1}\Sigma^\infty_{\mathbb{P}^1}X_+, n \geq 0$ is called the category of very effective spectra, denoted by $\mathcal{SH}^{\text{veff}}(S)$.
2. For any $k \in \mathbb{Z}$, the category of k-very effective spectra is generated by

$$\Sigma^n_{\mathbb{S}^1}\Sigma^\infty_{\mathbb{P}^1}\left(\left(\mathbb{P}^1\right)^{\wedge k} \wedge X\right)_+, n \geq 0.$$

where $X \in \mathcal{S}\mathrm{m}_S$ a smooth scheme.

the inclusion $\tilde{\imath}_k : \mathcal{SH}^{\text{veff}}(S)(k) \to \mathcal{SH}(S)$ we have a right adjoint $\tilde{r}_k : \mathcal{SH}(S) \to \mathcal{SH}^{\text{veff}}(S)(k)$, and this induces a tower of very effective cover of spectra:

4.1 Slice filtrations

$$\ldots \to \tilde{E}^{k+1} := \tilde{\iota}_{k+1}\tilde{r}_{k+1}E \to \tilde{E}^{k} \to \tilde{E}^{k-1} \to \ldots$$

called the generalized slice filtration of motivic spectra. The graded piece is called the generalized slice.

Remark

Voevodsky made several conjectures about slices. one of them conjectures that $\mathrm{gr}^0(\mathrm{KGL}) \simeq M_{\mathbb{Z}}$ representing the motivic cohomology.

4.1 Slice filtrations

The calculation of slices of MGL_k is very important for our purpose.

Theorem 4.1.4 (Hopkins-Morel-Hoyois) Let k be a field of exponential characteristic e and $\mathrm{MGL} \in \mathcal{SH}(k)$ the algebraic cobordism spectrum. The canonical map

$$f : \mathrm{MGL}/(b_1, b_2, \ldots)[1/e] \to M_{\mathbb{Z}}[1/e]$$

is an equivalence.

Corollary 4.1.4.1 Let S be an essentially smooth scheme over a base field k. The slices of MGL^* in $\mathcal{SH}(S)$ are given by

$$\mathrm{gr}^t \mathrm{MGL}^* \simeq \Sigma^{2t,t} HL_t[1/e]$$

4.1 Slice filtrations

where L_t is the t-th graded piece of L viewed as an Adams graded MU_*-module. In particular, for k a field of characteristic 0, the zeroth slice of MGL is $M_{\mathbb{Z}}$.

Corollary 4.1.4.2 For any Landweber exact spectrum $E \in \mathcal{SH}(k)$ (especially MGL_k), the slice filtration and generalized slice filtration of E agree.

4.2 Degression: Beilinson t-structure and Décalage

Theorem 4.2.1 (Beilinson, Ariotta, Antieau) Let $\mathcal{C}$ be a stable ∞-category with sequential limits. There is a canonical categorical equivalence of complete filtrations and coherent cochain complexes:

$$\mathrm{Fil}_c(\mathcal{C}) \simeq \mathrm{Ch}^\bullet(\mathcal{C})$$

which sends a complete filtration $\mathrm{F}^\star$ to a cochain complex C with $C^n \simeq \mathrm{gr}^n \mathrm{F}^\star[n]$.

Definition 4.2.2 Let $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ be a t-structure on a stable ∞ -category $\mathcal{C}$ with sequential limits. Consider the pointwise t-structure on $\mathrm{Ch}^\bullet(\mathcal{C})$, by the above categorical equivalence we

4.2 Degression: Beilinson t-structure and Décalage

can thus define a t-structure $\left(\mathrm{Fil}_c(\mathcal{C})^{\text{Б}}_{\geq 0}, \mathrm{Fil}_c(\mathcal{C})^{\text{Б}}_{\leq 0}\right)$ on $\mathrm{Fil}_c(\mathcal{C})$, which is called the Beilinson t-structure on $\mathrm{Fil}_c(\mathcal{C})$.

For incomplete filtrations $\mathrm{F}^\star \in \mathrm{Fil}(\mathcal{C})$, we can glue the trivial t-structure on $\mathcal{C}$ and Beilinson t-structure on $\mathrm{Fil}_c(\mathcal{C})$ to get the Beilinson t-structure on $\mathrm{Fil}(\mathcal{C})$.

Definition 4.2.3 Let $\mathrm{F}^\star$ be a filtration. Consider the Whitehead tower with respect to the Beilinson t-structure on $\mathrm{Fil}(\mathcal{C})$

$$\ldots \to \tau^{\text{Б}}_{\geq n+1}\mathrm{F}^\star \to \tau^{\text{Б}}_{\geq n}\mathrm{F}^\star \to \tau^{\text{Б}}_{\geq n-1}\mathrm{F}^\star \to \ldots$$

By taking the realization, we get a new filtered object of $\mathcal{C}$

4.2 Degression: Beilinson t-structure and Décalage

$$\dots \to |\tau^{\text{Б}}_{\geq n+1}\mathrm{F}^\star| \to |\tau^{\text{Б}}_{\geq n}\mathrm{F}^\star| \to |\tau^{\text{Б}}_{\geq n-1}\mathrm{F}^\star| \to \dots$$

This is called the décalage of $\mathrm{F}^\star$, and we denote $\mathrm{Dec}^\bullet(\mathrm{F}^\star)$. If $\mathrm{F}^\star$ is a filtration on X, Since we have natural maps $\tau^{\text{Б}}_{\geq n}\mathrm{F}^\star \to \mathrm{F}^\star$, we then have a map

$$|\tau^{\text{Б}}_{\geq n}\mathrm{F}^\star| \to |\mathrm{F}^\star| \to X$$

hence $\mathrm{Dec}^\bullet(\mathrm{F}^\star)$ is a filtration on X.

Example. (Berthelot-Ogus, Bhatt-Morrow-Scholze)
Let $M^\bullet \in \mathrm{Ch}(\mathrm{Mod}_A)$ and $f \in A$ such that M is f-torsion free. Consider the f-adic filtration $\mathrm{F}^\star$ on $M^\bullet$, we define a new complex by

$$\left(\eta_f M\right)^i = \{x \in \mathrm{F}^i M^i : d^i(x) \in \mathrm{F}^{i+1} M^{i+1}\}$$

4.2 Degression: Beilinson t-structure and Décalage

and this construction descends to $L\eta_f : D(A) \to D(A)$. In fact, $L\eta_f M$ can viewed as $\mathrm{Dec}^0(\mathrm{F}^\star M)$.

Now we study how to build spectral sequences out of filtrations.

A filtration $\mathrm{F}^\star$ gives rise to a coherent cochain complex

$$\dots \to \mathrm{gr}^{-s-1}\mathrm{F}^\star[-s-1] \to \mathrm{gr}^{-s}\mathrm{F}^\star[-s] \to \mathrm{gr}^{-s+1}\mathrm{F}^\star[-s+1] \to \dots$$

Apply the π_t functor and after suitable suspension yields a coherent cochain complex in the heart $\mathcal{C}^\heartsuit$ of $\mathcal{C}$

$$\dots \to \pi_{s+t+1}\mathrm{gr}^{-s-1}\mathrm{F}^\star \to \pi_{s+t}\mathrm{gr}^{-s}\mathrm{F}^\star \to \pi_{s+t-1}\mathrm{gr}^{-s+1}\mathrm{F}^\star \to \dots$$

Definition 4.2.4 The E_r-page of the spectral sequence associated to a filtration $\mathrm{F}^\star$ is defined inductively to be

4.2 Degression: Beilinson t-structure and Décalage

$$E_1^{s,t}(\mathrm{F}^\star) := \pi_{s+t}\mathrm{gr}^{-s}\mathrm{F}^\star$$

$$E_{r+1}^{s,t}(\mathrm{F}^\star) := E_r^{-t,s+2t}(\mathrm{Dec}(\mathrm{F}^\star))$$

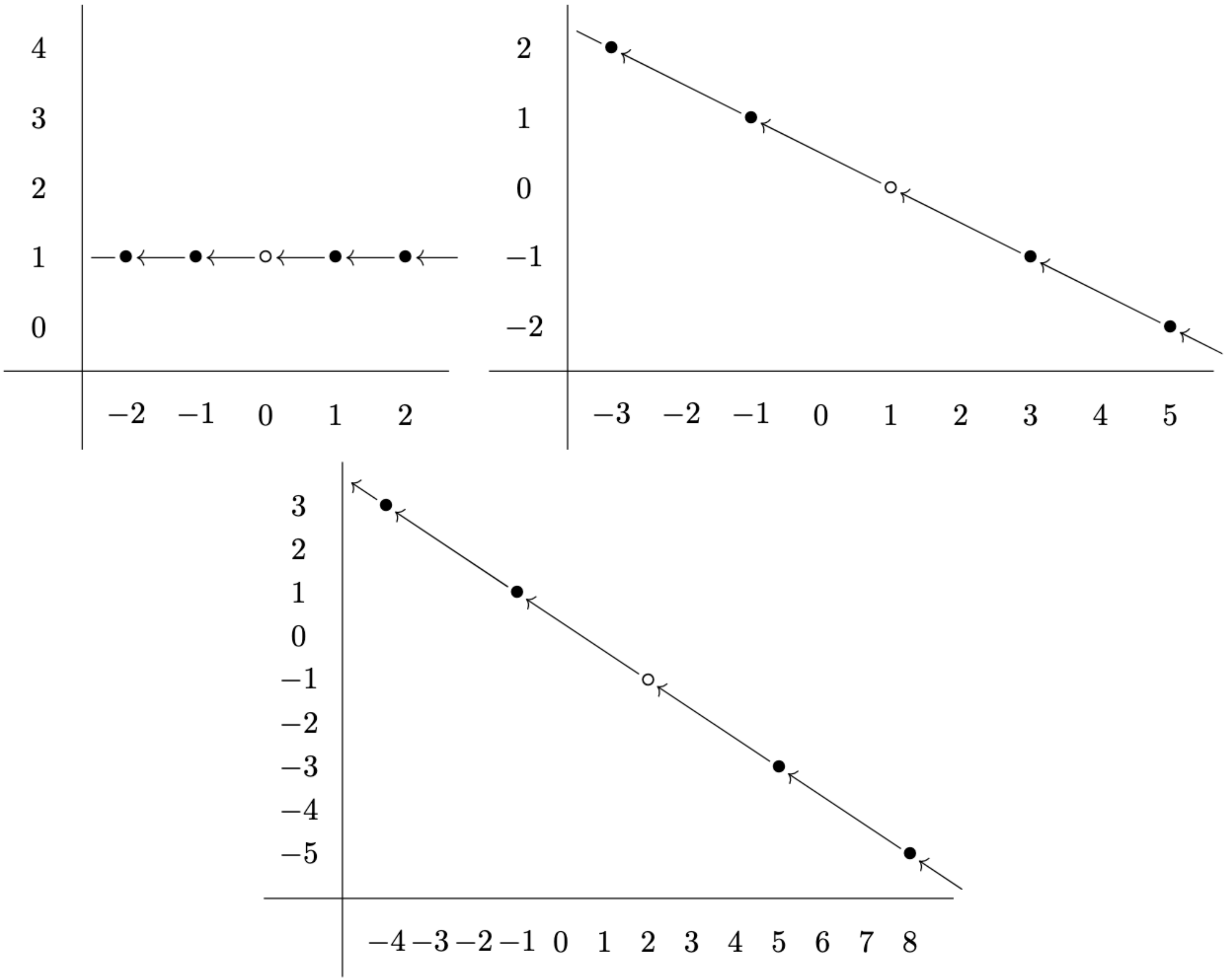
together with the differential:

$$d_1^{s,t} : \pi_{s+t}\mathrm{gr}^{-s}\mathrm{F}^* \to \pi_{s+t-1}\mathrm{gr}^{-s+1}\mathrm{F}^*$$

$$d_{r+1}^{s,t} := d_r^{-t,s+2t}.$$

4.2 Degression: Beilinson t-structure and Décalage

Visualization. (Antieau, 2024)



5. Realizations

5.1 Betti realizations

Realization functors are colimit preserving exact functors that send motivic spectra into a simpler stable ∞-categories, e.g. $\mathcal{SH}$.

Let k be a field of characteristic zero equipped with an embedding $k \hookrightarrow \mathbb{C}$.

1. We lift the functor $(-)(\mathbb{C}) : \mathcal{S}\mathrm{m}_k \to \mathrm{An}$, which associates a smooth scheme S with the homotopy type of the space of its complex points $S(\mathbb{C})$ under the analytic topology, to a colimit preserving functor $\mathrm{Re}_{\mathbb{C}} : \mathcal{P}(\mathcal{S}\mathrm{m}_k) \to \mathrm{An}$.
2. We can show this functor preserves Nisnevich excision and $\mathbb{A}^1$-invariance, thus $\mathrm{Re}_{\mathbb{C}} : \mathcal{P}(\mathcal{S}\mathrm{m}_k) \to \mathrm{An}$ factors through $\mathcal{H}(k)$ of motivic spaces. The same construction works for pointed spaces $\mathcal{H}(k)_*$.

5.1 Betti realizations

3. The final step is to post-compose the stabilization functor and check whether $\Sigma^\infty \circ \mathrm{Re}_{\mathbb{C}} : \mathcal{H}(k)_* \to \mathcal{SH}$ is symmetric monoidal and inverts $\mathbb{P}^1$, hence this functor factors through $\mathcal{SH}(k)$ and induces a functor $\mathrm{Re}_{\mathbb{C}} : \mathcal{SH}(k) \to \mathcal{SH}$, which is well-defined, symmetric monoidal, preserves colimits and finite products. We call this functor complex Betti realization.

4. The set $\mathbb{C}$-points of a $\mathbb{R}$-scheme naturally carries a $\mathbb{Z}/2$-action by complex conjugation, hence similarly, the functor $(-)(\mathbb{C})$ pre-composing with base change induces a functor $\mathrm{Re}_{\mathbb{R}} : \mathcal{H}(k) \to \mathcal{P}\left(\mathcal{O}_{\mathbb{Z}/2}\right)$, where $\mathcal{O}_{\mathbb{Z}/2}$ is the orbit category of $\mathbb{Z}/2$ and induces a functor $\mathrm{Re}_{\mathbb{R}} : \mathcal{SH}(k) \to \mathcal{SH}_{C_2}$, which is well-defined, symmetric monoidal, preserves colimits and finite products. We call this functor real Betti realization.

5.1 Betti realizations

Notice $\mathrm{Re}_{\mathbb{C}}$ preserves the $\mathcal{E}_n$-ring structure on $\mathcal{SH}(k)$ and $\mathcal{SH}$ for $1 \leq n \leq \infty$, this is a direct consequence of the fact that $\mathrm{Re}_{\mathbb{C}}$ is lax symmetric monoidal. Moreover we have

Proposition 5.1.1 The restriction of $\mathrm{Re}_{\mathbb{C}} : \mathcal{SH}^{\mathrm{veff}}(k)(m) \to \mathcal{SH}_{\geq 2m}$ is well-defined, where $\mathcal{SH}_{\geq 2m}$ is the subcategory of $2m$-connected spectra.

5.2 Étale realization

Let k be a field of characteristic p. Let $\ell \neq p$ be a prime.

Let $\mathrm{An}^\wedge$ be the category of profinite spaces, it is naturally identified with pro-objects of presheaves of Fin finite sets via the limit functor. For any shape $S \in \mathrm{Pro}(\mathrm{An})$, we can associate a profinite completion $S^\wedge \in \mathrm{An}^\wedge$ to it. Our target of étale realization functor will be the $\mathbb{S}^2$-stabilization of $\mathrm{An}^\wedge_*$, viewing $\mathbb{S}^2$ as a constant sheaf, to which we denote $\mathcal{SH}^{\wedge,\mathbb{S}^2}$.

We localize $\mathrm{Pro}(\mathrm{An})$ with respect to all pro-morphisms that induce isomorphisms on objectwise continuous cohomology with $\mathbb{Z}/\ell$-coefficients. We denote it as $\mathrm{Pro}(\mathrm{An})_\ell$.

There is a well-defined exact functor $\mathrm{Re}_{\text{ét}} : \mathcal{SH}(k) \to \mathrm{Pro}(\mathrm{An})_\ell \to \mathcal{SH}^{\wedge,\mathbb{S}^2}$. We call this functor étale realization.

5.3 Realization of Thom spectra

Theorem 5.3.1 Let MU denote the complex cobordism spectrum in $\mathcal{SH}$. Then there is an isomorphism of $\mathcal{E}_\infty$ rings: $\mathrm{Re}_{\mathbb{C}}\mathrm{MGL} \simeq \mathrm{MU}$, where MU is equipped with the $\mathcal{E}_\infty$ structure as a Thom spectrum.

The proof idea is essentially to identify the Betti realization of motivic Thom spectrum functor with the topological Thom spectrum functor and the following calculation:

Recall MU is identified with $M\left(\mathrm{BU} \xrightarrow{j} \mathcal{SH}^{\simeq}\right)$ and $\mathrm{MGL} = M_k\left(K^\circ \xrightarrow{j} \mathcal{SH}^{\simeq}\right)$, so we reduce to check $\mathrm{Re}_{\mathbb{C}}(K^\circ) \simeq \mathrm{BU}$. To see this, one compute

5.3 Realization of Thom spectra

$$\mathrm{Re}_{\mathbb{C}}(K^{\circ}) \simeq \mathrm{Re}_{\mathbb{C}}(\mathrm{BGL}) = \mathrm{Re}_{\mathbb{C}}(\mathrm{colim}_{\Delta^{\mathrm{op}}}(\mathrm{GL} \to \mathrm{GL} \times \mathrm{GL} \to \dots))$$

$$\underset{1}{\simeq} \mathrm{B}\ (\mathrm{Re}_{\mathbb{C}}(\mathrm{GL})) = \mathrm{B}\left(\bigcup_n \mathrm{GL}_n(\mathbb{C})\right) \underset{2}{\simeq} \mathrm{B}\left(\bigcup U_n(\mathbb{C})\right) = \mathrm{BU}$$

where 1 is true since $\mathrm{Re}_{\mathbb{C}}$ preserves colimit and 2 comes from the deformation retract $U_n(\mathbb{C}) \hookrightarrow \mathrm{GL}_n(\mathbb{C})$.

Corollary 5.3.1.1 Let MU^* be the Postnikov filtration on MU and MGL^* the slice filtration on $\mathrm{MGL} \in \mathcal{SH}(k)$, then $\mathrm{Re}_{\mathbb{C}}(\mathrm{MGL}^n)$ is $(2n-1)$-connected, and we have an isomorphism in $\mathcal{SH}$:

$$\tau_{\geq 2n}\mathrm{Re}_{\mathbb{C}}(\mathrm{MGL}^n) \cong \mathrm{MU}^{2n}$$

6. Spectral sequences

6.1 Atiyah-Hirzebruch spectral sequences

1. Let $E \in \mathcal{SH}(k)$ and E^* the slice filtration on E. Then Section 4.2 yields a spectral sequence related to E^*, we call this the motivic slice spectral sequence associated to E^*.
2. We are primarily interested in the global section of this slice spectral sequence. This is also what Levine is referring to as motivic Atiyah-Hirzebruch spectral sequence:

$$E_1^{s,t}(AH)(E) := \pi_{s+t,0}(\mathrm{gr}^s E^*)(\mathrm{Spec}\ k) \Rightarrow \pi_{s+t,0}(E)(\mathrm{Spec}\ k)$$

3. Our goal is to identify this spectral sequence with something out of topological nature. We shall do it in characteristic 0 and p-cases, where the first case can be divided into global and ℓ-local pictures.

6.2 Characteristic 0 case

Lemma 6.2.1 The Betti realization functor gives an isomorphism

$$\mathrm{Re}\left(\mathrm{gr}^{[a,b)}\mathbb{S}_k^{\star}\right) \cong \mathrm{gr}^{[2a,2b)}\mathrm{Dec}^{\bullet}\left(\mathrm{MU}^{\wedge *+1}\right)$$

Proof. We first notice the Postnikov tower on MU can be applied termwise on cosimplicial spectrum $s \mapsto \mathrm{MU}^{\wedge s+1}$ and incudes a filtration on it. (note here this filtration is indeed complete)

Since $\mathrm{Re}_{\mathbb{C}}$ is an exact symmetric monoidal functor, it preserves cofiber sequences and we have an isomorphism

$$\mathrm{Re}_{\mathbb{C}}\left(\mathrm{colim}_{s\leq N}\mathrm{Tot}_s\mathrm{gr}^{[a,b)}\mathrm{MGL}^{\wedge *+1}\right) \cong \mathrm{colim}_{s\leq N}\mathrm{Tot}_s\tau^{\text{Б}}_{\geq 2a}\tau^{\text{Б}}_{\leq 2b}\mathrm{MU}^{\wedge *+1}.$$

6.2 Characteristic 0 case

We still need to calculate the left side to get rid of the colimit. We observe this follows from the descendability of $\mathrm{gr}^{[a,b)}\mathbb{S}_k^* \to \mathrm{gr}^{[a,b)}(\iota_s)_*\mathrm{MGL}^{\wedge *+1}$: Indeed, descendability implies an equivalence

$$\mathrm{gr}^{[a,b)}\mathbb{S}_k^* \simeq \mathrm{Tot}_s \mathrm{gr}^{[a,b)}(\iota_s)_*\mathrm{MGL}^{\wedge *+1}$$

in $\mathcal{SH}(k) = \mathrm{Mod}_{\mathbb{S}_k}$. Now take $N = \infty$ and by definition of décalage functor, we have the desired result. □

Therefore, we reduce to show:

Lemma 6.2.2 Let $a \leq b \leq s+1$, denote $\iota_s^* : \Delta^{\leq s} \hookrightarrow \Delta$ the inclusion of simplicial sets and $(\iota_s)_*$ its right adjoint. Then $\mathrm{gr}^{[a,b)}\mathbb{S}_k^* \to \mathrm{gr}^{[a,b)}(\iota_s)_*\mathrm{MGL}^{\wedge *+1}$ is $\mathcal{E}_\infty$-descendable.

Proof. The unit map $c\mathbb{S}_k \to \mathrm{MGL}^{\wedge *+1}$ induces a map $\mathbb{S}_k \to \mathrm{holim}_{\Delta^{\leq s}}(\iota_s)_* \mathrm{MGL}^{\wedge *+1}$ since $\mathrm{holim}(\iota_s)_*\mathbb{S}_k \cong \mathbb{S}_k$. On the other hand, there is an equivalence

$$\mathrm{holim}_{\Delta^{\leq s}}(\iota_s)_* \mathrm{MGL}^{\wedge *+1} \simeq \mathrm{Tot}_s(\iota_s)_* \mathrm{MGL}^{\wedge *+1}$$

by definition. We now show $\mathrm{gr}^{[a,b)}\mathbb{S}_k^* \cong \mathrm{Tot}_s \mathrm{gr}^{[a,b)}(\iota_s)_* \mathrm{MGL}^{\wedge *+1}$, which, in light of equivalent definitions of descendability, suffices to construct a retraction from $\mathrm{Tot}^s \mathrm{gr}^{[a,b)}(\iota_s)_* \mathrm{MGL}^{\wedge *+1}$ to $\mathrm{gr}^{[a,b)}\mathbb{S}_k^*$.

We show $\mathrm{cofib}\left(\mathrm{gr}^{[a,b)}\mathbb{S}_k^* \to \mathrm{Tot}_s \mathrm{gr}^{[a,b)}(\iota_s)_* \mathrm{MGL}^{\wedge *+1}\right) = 0$. For this, notice that by Hopkins-Morel-Hoyois $\mathrm{gr}^0\mathbb{S}_k^* \cong \mathrm{gr}^0\mathrm{MGL}^* \simeq M_{\mathbb{Z}}$. Let $\overline{\mathrm{MGL}}$ be the cofiber of $\mathbb{S}_k \to \mathrm{MGL}$, then it follows that $\overline{\mathrm{MGL}}^1 = \overline{\mathrm{MGL}}$, thus $\left(\overline{\mathrm{MGL}}^{\wedge s+1}\right)^{s+1} = \overline{\mathrm{MGL}}^{\wedge s+1}$, and we have

$$\left(\Omega^s \overline{\mathrm{MGL}}^{\wedge s+1}\right)^{s+1} \cong \Omega^s \overline{\mathrm{MGL}}^{\wedge s+1}.$$

6.2 Characteristic 0 case

Hence for $a \le b \le s+1$

$$\mathrm{gr}^{[a,b)}\left(\Omega^s\overline{\mathrm{MGL}}^{\wedge s+1}\right)^{\star} = 0.$$

The following fiber sequence is well known [HA, §4.7.2]:

$$\Omega^s\overline{\mathrm{MGL}}^{\wedge s+1} \to \mathrm{Tot}_s\mathrm{MGL}^{\wedge *+1} \to \mathrm{Tot}_{s-1}\mathrm{MGL}^{\wedge *+1}$$

and after truncation to $(\iota_s)_*\mathrm{MGL}^{\wedge *+1}$ we have a cofiber sequence

$$\mathbb{S}_k \simeq \mathrm{Tot}_{s-1}(\iota_s)_*\mathrm{MGL}^{\wedge *+1} \to \mathrm{Tot}_s(\iota_s)_*\mathrm{MGL}^{\wedge *+1} \to \Omega^s\overline{\mathrm{MGL}}^{\wedge s+1}.$$

Then we conclude by taking the associated graded pieces. □

6.2 Characteristic 0 case

Proof of main theorem. Since a and b in the lemma are arbitrary, we have an isomorphism of the slice spectral sequences associated to two filtrations $\mathrm{Re}(\mathbb{S}_k^\star)$ and $\mathrm{Dec}^\bullet(\mathrm{MU}^{\wedge *+1})$, where all the odd homotopy groups of $\mathrm{MU}^{\wedge s+1}$ vanish.

On the other hand, the realization functor induces an isomorphism

$$\pi_{n,0}(\mathrm{gr}^{[a,b)}\mathbb{S}_k^\star)(k) \cong \pi_n(\mathrm{Re}(\mathrm{gr}^{[a,b)}\mathbb{S}_k^\star))$$

whence the first spectral sequence is just $E(AH)$ by definition. This, after a change of E_2-spectral sequence to E_1-spectral sequence, yields

$$E_r^{p,q}(AH) \cong E_{2r-1}^{2p,q-p}(\mathrm{Dec}(\mathrm{MU}^{\wedge *+1})) \cong E_{2r}^{2p,q-p}(\mathrm{Dec}(\mathrm{MU}^{\wedge *})).$$

Now by décalage:

$$E_{2r}^{2p,q-p}(\mathrm{Dec}(\mathrm{MU}^{\wedge *})) \cong E_{2r+1}^{3p+q,-2p}(\mathrm{MU}^{\wedge *}) = E_{2r+1}^{3p+q,2p}(AN).$$

6.3 Completion and localization

We can construct the motivic counterpart of the Brown-Peterson spectra, a direct summand of $\mathrm{MU}_{(\ell)}$, the Bousfield localization of MU. As this is a Landweber exact theory, we have

$$\mathrm{gr}^0\mathbb{S}_k^* \otimes \mathbb{Z}_{(\ell)} \simeq \mathrm{gr}^0\mathrm{MGL}^* \otimes \mathbb{Z}_{(\ell)} \simeq \mathrm{gr}^0\mathrm{BP}_{\mathrm{mot}}^{(l)}.$$

Apply the descendability argument to $\mathbb{S}_k^* \otimes \mathbb{Z}_{(\ell)} \to \mathrm{BP}_{\mathrm{mot}}^{(\ell)}$ we can establish an isomorphism of spectral sequences:

$$E_2^{s,t}(AN)_\ell = \mathrm{Ext}^{s,t}_{\mathrm{BP}_*^{(\ell)}(\mathrm{BP}^{(\ell)})}\left(\mathrm{BP}_*^{(\ell)}, \mathrm{BP}_*^{(\ell)}\right) \Longrightarrow \pi_{t-s}\mathbb{S} \otimes \mathbb{Z}_{(\ell)}$$

and

$$E_1^{p,q}(AH)_\ell = \pi_{-p-q,0}(\mathrm{gr}^{-q}\mathbb{S}_k^\star)(k) \otimes \mathbb{Z}_{(\ell)} \Longrightarrow \pi_{-p-q,0}(\mathbb{S}_k)(k) \otimes \mathbb{Z}_{(\ell)}.$$

6.3 Completion and localization

Similarly the completion with respect primes away from characteristic is well defined, for example

$$\mathrm{MGL}_{\ell}^{\wedge} := \lim_{n} \mathrm{MGL}/\ell^{n}.$$

In characteristic zero this is not well behaved under realizations because it is an infinite limit.

6.3 Completion and localization

Completion can also happen for other elements in ring spectra.

Definition 6.3.1

1. The algebraic Hopf map is the class $\eta \in \pi_{1,1}(\mathbb{S}_k)$ induced by the coordinate map

$$\mathbb{A}^2_S - \{0\} \to \mathbb{P}^1_S, (x, y) \mapsto [x : y].$$

2. The limit of the sequential tower

$$... \to E \wedge \operatorname{cofib}(\eta^{n+1}) \to E \wedge \operatorname{cofib}(\eta^n) \to E \wedge \operatorname{cofib}(\eta^{n-1}) \to ...$$

is called the η-completion $E^\wedge_\eta$ of E.

6.4 ℓ-adic case

We can define an étale version of stable motivic homotopy category, of which we denote $\mathcal{SH}_{\text{ét}}(S)$. This category turns out to be a non-full localization of $\mathcal{SH}(S)$. The identity functor id : $\mathcal{S}\mathrm{m}_S \to \mathcal{S}\mathrm{m}_S$ induces a geometric morphism of ∞-topoi

$$\varepsilon_* : \mathrm{Shv}^{\wedge}_{\text{ét}}(\mathcal{S}\mathrm{m}_S) \to \mathrm{Shv}_{\mathrm{Nis}}(\mathcal{S}\mathrm{m}_S)$$

with a left adjoint ε^*. This adjunction descends to

$$\varepsilon^* : \mathcal{SH}(S) \rightleftarrows \mathcal{SH}_{\text{ét}}(S) : \varepsilon_*$$

Since $H^{0,1}(\mathrm{Spec}\ k; \mathbb{Z}/n) \simeq \mu_n(k)$. Let ζ be a primitive n-th roots of unity in k, and let β_n be the associated element of $H^{0,1}(\mathrm{Spec}\ k; \mathbb{Z}/n)$. the spectral sequence

$$H^{p+2t,q+t}(k; \mathbb{Z}/n) \otimes L_t[1/p] \Longrightarrow \mathrm{MGL}^{p,q}(k)[1/p]$$

sends β_n to an element in $\mathrm{MGL}^{0,1}(k;\mathbb{Z}/n)$, which we call motivic Bott element. The element β_{ℓ^v} actually lives in $(\mathrm{MGL}/\ell^v)^{0,N}(k)$ for some N, and the formal inversion with respect to β_{ℓ^v} is independent on the choice of the root of unity ζ.

Proposition 6.4.1 (Elmanto et. al) For any $v \geq 1$ The unit of the adjunction induces an equivalence of spectra

$$\mathrm{MGL}/\ell^v[\beta_{\ell^v}^{-1}] \xrightarrow{\simeq} \mathrm{MGL}^{\text{ét}}/\ell^v$$

where $\mathrm{MGL}^{\text{ét}} := \varepsilon_*\varepsilon^*(\mathrm{MGL})$ the étale localization of algebraic cobordism.

6.4 ℓ-adic case

Theorem 6.4.2 There is an isomorphism of graded abelian groups induced by étale realization:

$$\left(\bigoplus_{p,q} \mathrm{MGL}^{p,q}(k) \otimes \mathbb{Z}_\ell\right)[\beta^{-1}] \cong \bigoplus_{p} (\mathrm{MU}_\ell^\wedge)^p[\beta^{-1}]$$

where β is the collection of β_{ℓ^v} for all $v \geq 1$.

More generally, this isomorphism works for all Landweber exact theories, especially the symmetric products of MGL. We thus can prove a complete version of our main theorem and this yields:

$$\left(\pi_{s+t,0}(\mathbb{S}_k[1/p])(k)\right)_\ell^\wedge[\beta^{-1}] \cong (\pi_{s+t}\mathbb{S})_\ell^\wedge[\beta^{-1}]$$

7. Future work

7.1 Even filtrations

Theorem 7.1.1 (Pstrągowski, Gheorghe et. al)

$$\mathcal{SH}^{\mathrm{cell}}(\mathbb{C})_p^\wedge \simeq \mathrm{Syn}_p^\wedge \simeq \mathrm{Mod}_{\Gamma_* \mathbb{1}}$$

where Syn represents the category of synthetic spectra, which is a way to encode the Adams-Novikov spectral sequence using a one-parameter deformation of $\mathcal{SH}$.

Since MGL is cellular, this equivalence reveals that over $\mathbb{C}$, the behavior of MGL-modules should be purely topological under some mild finiteness conditions. In fact, the étale case suggests that more should be true over arbitrary algebraic closed fields, though we don't know how to precisely state that, since cellularity is not closed under infinite limits, in particular, completions.

7.2 p-adic case

The étale realization is not well defined in p-adic case since it is not $\mathbb{A}^1$-invariant. There are some possible ways to solve this.

1. Use a non-$\mathbb{A}^1$-invariant motivic homotopy theory, e.g. logarithmic motivic homotopy or $\mathbb{P}^1$-invariant spectra.
2. Use an intermediate topology to realize, e.g. tame site. This approach needs us to find an alternative to Elmanto et. al's construction, especially an alternative of Bott element.

Thank you for listening!