Logarithmic de Rham-Witt sheaf III

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1 The category $\mathrm{VDR}(X)$

The construction of de Rham-Witt complex roots from the construction of $W_{\bullet}\mathcal{O}_X$ as a projective system. Essentially we want a system of differential graded algebras $\left\{W_n\Omega^*_{X/S}\right\}_{n\geq 0}$ such that there are families of natural maps (compatible with differentials) $F : W_{n+1} \to W_n$ and $V : W_n \to W_{n+1}$ just like in the theory of Witt vectors.

Inspired by this demand, we define a category of V-framed complexes first, and define the complex without Frobenius. We can prove certain universality of this complex. Finally, we try to lift the Frobenius onto the de Rham-Witt complex.

Generally we let X be a Grothendieck topos, for example $\mathrm{Shv}\big((\mathrm{Perf}_S)_{\text{ét}}, \mathbb{F}_p\big)$ in the previous consideration.

Definition 1.1 A de Rham V-pro-complex on X consists of an inverse system $(M_n, \mathrm{Res}_n : M_{n+1} \to M_n)_{n\in\mathbb{Z}}$, of ($\mathbb{Z}$-linear) differential graded algebras of X and a family of additive maps $(V^i_n : M^i_n \to M^i_{n+1})_{i,n\in\mathbb{Z}}$, called Verschiebung maps such that

1. $\mathrm{Res} \circ V = V \circ \mathrm{Res}$.
2. $M_n = 0$ for $n \leq 0$. M^0_1 is an $\mathbb{F}_p$-algebra and $M^0_n = W_n(M^0_1)$ for $n \geq 1$. On M^0_n, Res and V will be the usual restriction and Vershiebung map of Witt vectors.
3. $V(xdy) = (Vx)dVy$.
4. For $x \in M^0_1$, let $[x]$ be the Teichmüller lift of x, then for any $y \in M^0_n$ we have

$$(Vy)d[x] = V\big([x]^{p-1}y\big)dV[x]. \tag{1.1}$$

Remark 1.2

1. Following [BLM21], we denote the inverse limit of this pro-complex as M_∞. Notice that in [BLM21] we don't require M^0_∞ to be precisely the Witt ring of M^0_1, but just a canonical map $\beta : W(M^0_1) \to M^0_\infty$ to it.
2. By induction on the third condition (together with the second condition), we can show that every section in M^i_n is locally additively generated by $adx_1...dx_i$ for $a, x_1, ..., x_i \in M^0_n$.

Definition 1.3 A morphism between de Rham V-pro-complexes on X is a family of map $(f_n : M_n \to N_n)$ compatible with differentials, restrictions and Verschiebungs such that $f^0_n = W_n(f^0_1)$ for all $n \geq 1$. This defines a category of de Rham V-pro-complexes on X, denoted by $\mathrm{VDR}(X)$.

2 The de Rham-Witt complex $W_\bullet\Omega^*_X$

We know construct the de Rham-Witt complex, first by a categorical approach, then by some explicit formulas.

Theorem 2.1 *[[Ill79], I.1.3] The functor*

$$\mathrm{VDR}(X) \to \mathrm{Alg}_{\mathbb{F}_p}(X), \{M_n\} \mapsto M_1^0 \tag{2.1}$$

has a left adjoint $A \mapsto W_\bullet\Omega^*_A$.

Proof. A collection of differential graded ideals $\left\{I_n \subset \Omega_{W_nA}\right\}$ calls good if

1. The restriction map $R : \Omega_{W_{n+1}A} \to \Omega_{W_nA}$ carries I_{n+1} to I_n.
2. The Verschiebung map V over W_nA induces a map $V : \Omega^k_{W_nA}/I^k_n \to \Omega^k_{W_{n+1}A}/I^k_{n+1}$ such that
$$V(adx_1 ... dx_k) = VadVx_1 ... dVx_k. \tag{2.2}$$
3. For each $x \in A$ and $y \in W_nA$, the difference
$$Vyd[x] - V([x]^{p-1}y)dV[x] \in I_{n+1}. \tag{2.3}$$

It's easy to see there is a minimal differential graded ideal satisfy these three conditions. In fact, just take the graded ideal generated by the second and the third condition. We shall define

$$W_n\Omega_A := \Omega_{W_nA}/I_n. \tag{2.4}$$

Clearly there is a surjection $\Omega_{W_nA} \to W_n\Omega_A$ and the universal property follows from this surjection, the minimality of I_n and the composition with the universal property of de Rham complex. ■

Definition 2.2 For A an $\mathbb{F}_p$-algebra on X, we call the complex $W\Omega_A := W_\infty\Omega_A = \lim_\leftarrow W_n\Omega_A$ the de Rham-Witt complex of A. Specially if X is a ringed topos, then we write $W_\bullet\Omega_X$ and $W\Omega_X$ for $W_\bullet\Omega_{\mathcal{O}_X}$ and $W\Omega_{\mathcal{O}_X}$ respectively.

Remark 2.3

1. In the proof we see that the restriction $\mathrm{Res} : W_{n+1}\Omega_X \to W_n\Omega_X$ is in fact surjective, therefore we can view each $W_n\Omega_X$ as a quotient of $W\Omega_X$.
2. The functor W_n commutes with filtered colimits, by abstract nonsense we see that $\mathrm{VDR}(X)$ has filtered colimits and
$$\lim_{\to} W_\bullet\Omega_{A_i} \cong W_\bullet\Omega_{\lim_\to A_i}. \tag{2.5}$$
3. Similarly the functor W_n commutes with Γ, hence we have
$$W_\bullet\Omega_{\Gamma(U,A)} \cong \Gamma(U, W_\bullet\Omega_A) \tag{2.6}$$

   for U an open over X.

We now extract some explicit information from the proof of Theorem 2.1.

Corollary 2.4 *For all $n \geq 1$, the homomorphism of differential graded algebras $\pi_n : \Omega_{W_n(A)} \to W_n\Omega_A$ is surjective and in particular, π_1 is an isomorphism.*

For A a perfect $\mathbb{F}_p$-algebra, we have $\Omega^i_{W_n(A)} = 0$ for all $i > 0$, by the previous corollary we thus have

Corollary 2.5 *For A a perfect $\mathbb{F}_p$-algebra, $W_n\Omega^i_A = 0$ for $i > 0$.*

We now show some functorialities of the construction. For $f : X \to Y$ a morphism of ringed topos, there are induced morphisms in VDR:

$$\begin{gathered} W_\bullet\Omega_Y \to f_* W_\bullet\Omega_X \\ f^{-1}W_\bullet\Omega_Y \to W_\bullet\Omega_X. \end{gathered} \tag{2.7}$$

Clearly if $f^{-1}\mathcal{O}_Y \cong \mathcal{O}_X$, the second map is an isomorphism. In particular, we have

$$(W_\bullet\Omega_X)_x \cong W_\bullet\Omega_{X,x}. \tag{2.8}$$

On the other hand, suppose f is étale, then we have the following theorem.

Theorem 2.6 *Let $f : X \to Y$ be an étale map, then we have an $W_n\mathcal{O}_X$-linear isomorphism for all n:*

$$f^* W_n\Omega_Y \overset{\cong}{\to} W_n\Omega_X. \tag{2.9}$$

Proof. We explain the linearity first. For $A \to B$ a map of $\mathbb{F}_p$-algebras, functoriality defines a map $W_\bullet\Omega_A \to W_\bullet\Omega_B$. In particular, the composition with

$$W_n\mathcal{O}_X \to \Omega^i_{W_n\mathcal{O}_X} \overset{\pi}{\to} W_n\Omega^i_X \to f^* W_n\Omega^i_Y \tag{2.10}$$

defines $W_n\Omega_Y$ as a $W_n\mathcal{O}_X$-linear differential graded algebra.

Now since the assertion is local, we may assume X and Y affine with associated ring B and A. We turn this into a Zariski local question by the following lemma:

Lemma 2.7 *For $n \geq 1$ and all $i \in \mathbb{Z}$, $W_n\Omega^i_X$ is a quasi-coherent sheaf on $W_n(X)$, the infinitesimal extension of X. For each affine open $U = \operatorname{Spec}(A)$ we have $\Gamma(U, W_n\Omega^i_X) = W_n\Omega^i_A$.*

Proof. On each affine open $X = \operatorname{Spec}(A)$, the constant sheaf A_X exhibits $\mathcal{O}_X$ as a localization. To given a W_nA-module structure on $W_n\Omega_X$, it suffices to prove

$$W_n\mathcal{O}_X \otimes_{W_nA_X} W_n\Omega^i_{A_X} \to W_n\Omega^i_X \tag{2.11}$$

is an isomorphism, this follows from the same trick below and we may leave the detail to reader. (Notice one has to deal with the localization of Witt vectors.) ■

Inspired by the last lemma, it suffices to proof

$$W_nB \otimes_{W_nA} W_n\Omega^i_A \to W_n\Omega^i_B \tag{2.12}$$

is an isomorphism. Since $W_nA \to W_nB$ is étale [[Ill79], 0.1.5.8], the differential d over $W_n\Omega_A$ naturally extends to a differential on $W_nB \otimes_{W_nA} W_n\Omega_A$ via $d(b \otimes x) = (db)x + b \otimes dx$ where db is the image of π in $W_n\Omega^1_A$. Hence the left side is also a differential graded algebra. The cocartesian diagram

$$\begin{tikzcd} W_nA \arrow[r, "F"] \arrow[d] & W_nA \arrow[d] \\ W_nB \arrow[r, "F"] & W_nB \end{tikzcd}$$

defines also a unique map $V : W_nB \otimes_{W_nA} W_n\Omega^i_A \to W_{n+1}B \otimes_{W_{n+1}A} W_{n+1}\Omega^i_A$. One can easily check this makes $W_\bullet B \otimes_{W_\bullet A} W_\bullet\Omega_A$ a de Rham V-pro-complex in $\mathrm{VDR}(B)$. Since $W_n\Omega_B$ is initial we have a map

$$W_n\Omega^i_B \to W_nB \otimes_{W_nA} W_n\Omega^i_A \tag{2.13}$$

and as π_B is surjective, this is an isomorphism. ■

3 The Vershiebung V and the Frobenius F

We start to define a Frobenius on de Rham-Witt complex. Since we only included V in the data of the category, we are forced to find a natural candidate of ring such that defining Frobenius is easy and the ring itself is universal enough to lift Frobenius to every de Rham-Witt complex.

Notice this complicate computation is the direct motivation for [BLM21], where they included the Frobenius as a part of the data of Dieudonné complex and require V by the formular $VF = p = FV$. (s. [[BLM21], Remark 1.2.8])

The new theory is, though limitted to the context of $\mathbb{F}_p$-algebras, whereas Illusie's approach works for any topos (after a suitable modification), for example, the big de Rham-Witt complex defined in [Hes15].

Proposition 3.1 *Let A be an $\mathbb{F}_p$-algebra on X, then:*
1. $xVy = V(FR(x)y)$ for all $x \in W_n(A)$ and $y \in W_{n-1}\Omega^i_A$.
2. $(d[x])Vy = V(([x]^{p-1}d[x])y)$ for all $x \in A$ and $y \in W_{n-1}\Omega^i_A$.

Proof. We first notice 2 is a consequence of the definition of de Rham V-pro complex. For 1, from the surjectivity of $\pi_{n-1} : \Omega_{W_{n-1}(A)} \to W_{n-1}\Omega_A$ we may assume $y = adx_1...dx_i$ where $a, x_1, ..., x_i \in W_{n-1}(A)$. Induction on i, when $i = 0$, this is just the property of Witt vectors. when $i = 1$, this is property of VDR (we view F(x)a) as a whole coefficient.) ■

Passing to the limit, we immediately have the following:

$$\begin{gathered} xVy = V(F(x)y), x \in W\mathcal{O}_X, y \in W\Omega_X^i \\ V(xdy) = (Vx)dVy, x, y \in W\Omega_X \\ d[x]Vy = V([x]^{p-1}d[x]y), x \in \mathcal{O}_X, y \in W\Omega_X. \end{gathered} \tag{3.1}$$

We now search for the generalization of the formulas above. The main obstruction will thus be define a reasonable Frobenius on the complex $W_\bullet\Omega_X$.

The suitable candidate for explicitly writing down the formula of Frobenius will be over the ring

$$C = \lim_{\substack{\rightarrow \\ i}} \mathbb{Q}_p\left[T_1^{p^{-i}}, ..., T_r^{p^{-i}}, T_{r+1}^{\pm p^{-i}}, ..., T_{r+s}^{\pm p^{-i}}\right] \tag{3.2}$$

where we can simply define $F\left(T^{p^{-i}}\right) = T^{p^{-i+1}}$.

Notice that $d\left(T^{p^{-i}}\right) = p^{-i}T^{p^{-i}}d\log T$ implies any section $x \in \Omega_{C/\mathbb{Q}_p}^m$ can be written uniquely as

$$x = \sum p_{i_1...i_m}(T)d\log T_{i_1}...d\log T_{i_m} \tag{3.3}$$

where $p_{i_1 \cdots i_m}(T)$ is a polynomial (after multiply a common factor) with coefficients in $\mathbb{Q}_p$.

Definition 3.2 Let

$$E^m := \left\{x \in \Omega_C^m : p_{i_1...i_m} \in \mathbb{Z}_p[T]\right\} \tag{3.4}$$

and $E = \{E^m\}$ be the subcomplex of Ω_C, called the complex of intergral forms. Clearly E is also a subalgebra of Ω_C as a differential graded algebra.

Now we lift F to Ω_C by acting on coordinates, let $V = pF^{-1}$ be the Verschiebung. Notice this definition satisfies the formulas in Definition 1.1 and Proposition 3.1 and we have $dF = pFd$ and $Vd = pdV$.

Let $A := \mathbb{F}_p\left[T_1, ..., T_r, T_{r+1}^{\pm 1}, ..., T_{r+s}^{\pm 1}\right]$ and $B := \mathbb{Z}_p\left[T_1, ..., T_r, T_{r+1}^{\pm 1}, ..., T_{r+s}^{\pm 1}\right]$. By definition (with some calculations) $E^0 = \sum_{n \geq 0} V^n B$ and the canonical lift of $B \to W(A)$ extends uniquely to an injective $\mathbb{Z}_p$-algebra map $\tau : E^0 \to W(A)$, commutes with Verschiebung and we have

$$E^0/V^r E^0 \simeq W_r(A). \tag{3.5}$$

(The idea is, we take the perfection of A and B and show $E^0 \subset B^{\text{perf}} \to W\left(A^{\text{perf}}\right)$, giving the desired morphism.)

Starting from this base case, we define a family of differential graded ideals

$$\mathrm{Fil}^r E^i := V^r E^i + dV^r E^{i-1} \tag{3.6}$$

and let $E_r := E/\mathrm{Fil}^r E$. Since $V(\mathrm{Fil}^r E) \subset \mathrm{Fil}^{r+1} E$ and for F vice versa, we have the induced maps

$$\begin{aligned} V &: E_r \to E_{r+1} \\ F &: E_{r+1} \to E_r. \end{aligned} \tag{3.7}$$

Illusie shows:

Theorem 3.3 *[[Ill79], I.2.5]*
1. $\{E_r\}$ together with V is an object in $\mathrm{VDR}(A) := \mathrm{VDR}(\mathrm{Shv}((\mathrm{Spec} A)_{\text{ét}}, \mathrm{Set}))$.
2. $W_\bullet \Omega_A \cong E$.

This is essentially the hardcore calculations we need. From this we can have explicit constructions of $W_\bullet \Omega_{\mathbb{G}_a}$ with Frobenius. What left to do is just some homological algebra and abstract nonsense.

Theorem 3.4 *[[Ill79], I.2.17] Let X be a ringed topos of $\mathbb{F}_p$-algebras, the Frobenius homomorphism $F : W_n \mathcal{O}_X \to W_{n-1} \mathcal{O}_X$ extends uniquely to a homomorphism $F : W_n \Omega_X \to W_{n-1} \Omega_X$ such that:*
1. For $n \geq 2$ and $x \in \mathcal{O}_X$, let $[x]_n \in W_n \mathcal{O}_X$ be the multiplicative representative, then

$$Fd[x]_n = [x]_{n-1}^{p-1} d[x]_{n-1}. \tag{3.8}$$

2. For $n \geq 1$ we have

$$FdV = d : W_n \mathcal{O}_X \to W_n \Omega_X^1. \tag{3.9}$$

Sketch of Proof. The uniqueness follows from the surjection π and the fact that $x \in W_n \mathcal{O}_X$ can be written as

$$x = [x_0]_n + V[x_1]_{n-1} + \ldots + V^{n-1}[x_{n-1}]_1. \tag{3.10}$$

To construct such an F, we define a map $Fd : W_n \mathcal{O}_X \to W_{n-1} \Omega_X^1$ by

$$Fdx = [x_0]_{n-1}^{p-1} d[x_0]_{n-1} + d[x_1]_{n-1} + \ldots + dV^{n-2} d[x_{n-1}]_1. \tag{3.11}$$

Via the map $FR : W_n \mathcal{O}_X \to W_{n-1} \mathcal{O}_X$, we can make $W_{n-1} \Omega_X^1$ a $W_n \mathcal{O}_X$-module and Fd is a derivation. To see this, in light of Remark 2.3 it suffices to assume X is finite type, thus a quotient of $A := \mathbb{F}_p[T_1, \ldots, T_n]$. By lifting x into $W_n A$. Now the explicit formula helps and we can verify.

Let $F : \Omega_{W_n \mathcal{O}_X} \to W_{n-1} \Omega_X$ be the universal homomorphism induced by the derivation Fd (and FR in degree 0), by construction F satisfies condition 1 and 2. Inspect the proof of Theorem 2.1, it suffices to show $F(I_n) = 0$, where I_n the minimal good differential graded ideal. It is amount of proving $Fz = 0$ for z the following four forms:

1. $z = \sum VadVx_1 ... dVx_k$ for $a, x_1, ..., x_k \in W_{n-1}\mathcal{O}_X$ such that $\sum adx_1 ... dx_k = 0 \in W_{n-1}\Omega_X$.
2. $z = \sum dVadVx_1 ... dVx_k$ as in 1.
3. $z = Vyd[x]_n - V\left([x]_{n-1}^{p-1} y\right) dV[x]_{n-1}$ where $x \in \mathcal{O}_X$ and $y \in W_{n-1}\mathcal{O}_X$.
4. $z = dVyd[x]_n - dV\left([x]_{n-1}^{p-1} y\right) dV[x]_{n-1}$ as in 3.

Now the first two types vanish following from condition $FdV = d$ and the last two vanish following from condition $Fd[x]_n = [x]_{n-1}^{p-1} d[x]_{n-1}$. This completes the proof. ■

Now we can check

$$FV = p = VF : W_n\Omega_X^i \to W_n\Omega_X^i \tag{3.12}$$

$$dF = pFd : W_n\Omega_X^i \to W_{n-1}\Omega_X^{i+1}, Vd = pdV : W_n\Omega_X^i \to W_{n+1}\Omega_X^{i+1} \tag{3.13}$$

$$FdV = d : W_n\Omega_X^i \to W_n\Omega_X^{i+1} \tag{3.14}$$

$$xVy = V(F(x)y), x \in W_n\Omega_X^i, y \in W_{n-1}\Omega_X^j \tag{3.15}$$

$$Fdx = R(x^{p-1}dx) + d\alpha(x), x \in W_n\mathcal{O}_X \tag{3.16}$$

where $\alpha : W_n\mathcal{O}_X \to W_{n-1}\mathcal{O}_X$ is the unique map such that $Fx = x^p + p\alpha(x)$.

Note it suffices to verify these maps for $A = \mathbb{F}_p[T_1, ..., T_N]$. The proof of Theorem 3.3 guarantees the construction in the previous theorem agrees with our first attempt of defining Frobenius, thus giving the first four formulas. For the last equation, we have

$$dFx = px^{p-1}dx + pd\alpha(x) \tag{3.17}$$

but as $dF = pFd$ and $W\Omega_X$ is torsion-free (one must check!), we have the formula after projection $W\Omega_X \to W_{n-1}\Omega_X$.

We are ready to define a higher analog of $v(r)$ in the previous talk. Since we have a Frobenius on $W\Omega_X$, the following definition makes sense.

Definition 3.5 For X a smooth scheme over a perfect $\mathbb{F}_p$-algebra, the logarithmic de Rham-Witt complex is defined to be the fixed point of Frobenius:

$$W\Omega_{X,\,\log} := \ker\left(W\Omega_X \overset{1-F}{\to} W\Omega_X\right). \tag{3.18}$$

Bibliography

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