Logarithmic de Rham-Witt sheaf II

Cheni Yuki Yang

For S a perfect scheme of characteristic p, we let Perf_S be the category of perfect S-schemes. Note that the forgetful functor $\mathrm{Perf}_S \to \mathrm{Sch}_S$ has a left adjoint by $(\cdot)^{\mathrm{pf}} : \mathrm{Sch}_S \to \mathrm{Perf}_S$, called the perfection of a scheme.

1 (derived) Duality of $v(r)$

Proposition 1.1 *For $L \in D^b(S)$ a bounded above complex of quasi-coherent $\mathcal{O}_S$-modules, there is a perfect paring*

$$L \otimes_{D\left((\mathrm{Perf}_S)_{\text{ét}}, \mathbb{F}_p\right)} L^\vee[-1] \overset{\cong}{\to} \mathbb{Z}/p. \tag{1.1}$$

where $\otimes$ is the derived tensor product.

Proof. Viewing L as a complex of étale sheaves of $\mathbb{F}_p$-modules since L is coherent, then L is perfect, so L is dualizable in $D(S_{\text{ét}}, \mathbb{F}_p)$ and we denote the dual by L^*. We show $L^* \cong L^\vee[-1]$

Now the Artin-Schreier sequence $0 \to \mathbb{Z}/p \to \mathcal{O}_S \overset{1-F}{\to} \mathcal{O}_S \to 0$ yields an identification in $D\Big((\mathrm{Perf}_S)_{\text{ét}}, \mathbb{F}_p\Big)$:

$$L^* \simeq \mathrm{cofib}\Bigg(R\mathrm{Hom}(L, \mathcal{O}_S) \overset{(1-F)_*}{\to} R\mathrm{Hom}(L, \mathcal{O}_S)\Bigg)[-1]. \tag{1.2}$$

Next we identify $R\mathrm{Hom}(L, \mathcal{O}_S)$. We may assume $L = \mathcal{O}_S$. Since étale sheaves are subcanonical and $\mathcal{O}_S$ is represented by $\mathbb{G}_a$, we are ready to classify $\mathrm{Hom}_{\mathbb{F}_p}(\mathbb{G}_a, \mathbb{G}_a)$, which is $\mathcal{O}_S[F]$ by [[DGB70], VIIA.4.2.1], thus globally we get an equivalence

$$R\mathrm{Hom}_{\mathrm{Shv}\left((\mathrm{Perf}_S)_{\text{ét}}, \mathbb{F}_p\right)}(L, \mathcal{O}_S) \cong \bigoplus_{n \geq 0} R\mathrm{Hom}_{\mathcal{O}_S}\big((F^n)^* L, \mathcal{O}_S\big). \tag{1.3}$$

As S is perfect, F is a finite isomorphism, therefore $F^!$ exists and is equivalent to F_*. But on $\mathcal{O}_S$ the pushforward of an isomorphism induces an isomorphism of $\mathbb{F}_p$-modules. We can now use the well-known Verdier duality result on perfect complexes to show

$$R\mathrm{Hom}_{\mathcal{O}_S}\Big((F^n)^* L, (F^n)^! \mathcal{O}_S\Big) \cong (F^n)^! R\mathrm{Hom}_{\mathcal{O}_S}(L, \mathcal{O}_S) \cong F^n_* L^\vee \cong L^\vee. \tag{1.4}$$

We just need to compute the cofiber of (1 - F)-endomorphism induced on $\bigoplus_{n \geq 0} L^\vee$.

F acts on $(x_0, x_1, ...,)$ by shifting by construction, hence

$$(1 - F)(x_0, x_1, ...,) = (x_0, x_1 - x_0, ...,) \tag{1.5}$$

and an easy calculation shows

$$0 \to \bigoplus_{n \geq 0} L^\vee \overset{1-F}{\to} \bigoplus_{n \geq 0} L^\vee \overset{\sum}{\to} L^\vee \tag{1.6}$$

is a cofiber sequence. Hence the cofiber of $1-F$ is concentrated on degree zero and is quasi-isomorphic to $L^\vee$. ■

Let $\pi : X \to S$ be a proper smooth morphism whose fibers have dimension m. We have the following duality result:

Warning: Milne's proof contains a gap, it relies on $v(r)$ to be a locally free coherent O_X-modules, which is not true. The correct argument can be found in [[Ber81], Théorème 3.5].

Theorem 1.2 *[[Mil76], Theorem 2.4] There is a quasi-isomorphism of complexes*

$$R\pi_* v(r) \to R\mathrm{Hom}(R\pi_* v(m-r), \mathbb{Z}/p) \tag{1.7}$$

induced by the paring $v(r) \times v(m-r) \to v(r)$ and the trace map

$$\eta : R\pi_* v(m) \to \mathbb{Z}/p. \tag{1.8}$$

Proof. The trace map was explained in the previous talk, as the composition

$$\eta : R\pi_* v(m) \to R\pi_* \mathrm{fib}\left(\Omega^m_{X/S} \overset{1-C}{\to} \Omega^m_{X/S}\right) \to \mathrm{fib}\left(\mathcal{O}_S \overset{1-F^{-1}}{\to} \mathcal{O}_S\right) \to \mathbb{Z}/p. \tag{1.9}$$

We show

$$R\pi_* v(r) \simeq R\mathrm{Hom}(R\pi_* v(m-r), R\pi_* v(m)). \tag{1.10}$$

Now as $v(r) = \mathrm{fib}\left(\Omega^r_{X/S,d=0} \overset{1-C}{\to} \Omega^r_{X/S}\right)$, it suffices to show

$$\begin{aligned} R\pi_* \Omega^r_{X/S,d=0} &\overset{\simeq}{\to} R\mathrm{Hom}_{\mathbb{F}_p}\left(R\pi_* \Omega^{m-r}_{X/S,d=0}, \mathbb{Z}/p\right)[1] \\ R\pi_* \Omega^r_{X/S} &\overset{\simeq}{\to} R\mathrm{Hom}_{\mathbb{F}_p}\left(R\pi_* \Omega^{m-r}_{X/S}, \mathbb{Z}/p\right)[1] \end{aligned} \tag{1.11}$$

and some commutativities(which we handled last time). This is clear from last proposition and the Grothendieck duality

$$R\pi_* \Omega^m_{X/S} \simeq \mathcal{O}_S. \tag{1.12}$$

(We claim $R\pi_* v(r)$ is bounded in $D(\mathrm{Shv}_{\text{ét}}(X, \mathbb{F}_p))$, this is because π is smooth hence étale and of relative dimension m.) ■

In fact, the [1] shift from Artin-Schreier sequence is the reason why we have a shifting of the duality of cohomology, as in the last time. On the derived version it vanishes, give a "perfect" duality.

From the identity

$$R\pi_* v(1) \cong R\pi_* Rf_* \mu_p \tag{1.13}$$

where $f : X_{\text{fl}} \to \left(\mathrm{Perf}_{X/S}\right)_{\text{ét}}$ the canonical site map, we immediately have the following isomorphism

$$R(\pi f)_*\mu_p \cong R\mathrm{Hom}\big(R(\pi f)_*\mu_p, \mathbb{Z}/p\big) \tag{1.14}$$

for $m = 2$, e.g., a smooth proper surface over $\mathbb{F}_p$.

Recall there is a classification theorem for perfect algebraic unipotent group schemes over characteristic p: each group has a composition series with quotient $\mathbb{G}_a$ or $\mathbb{Z}/p$. By the first proposition this yield an autoduality for all such group schemes. Combinin this with the previous theorem we have the following.

Corollary 1.3

1. *The étale sheaf $R^i\pi_*v(r)$ is represented by a perfect unipotent group scheme $G^i(r)$.*
2. *Let $U^i(r)$ be the connected component of $G^i(r)$, then $U^i(r)$ is O_S-linearly dual to $U^{m+1-i}(m-r)$.*
3. *Let $D^i(r) := G^i(r)/U^i(r)$, then there is a perfect paring of étale group scheme*

$$D^i(r) \times D^{m-i}(m-r) \to \mathbb{Z}/p. \tag{1.15}$$

2 Iterated Cartier operator

We now turn our attention to generalized the fundamental exact sequence to higher dimension. We introduce the iterated Cartier operator over a (perfect) $\mathbb{F}_p$-scheme S first.

Let $X \to S$ be a smooth S-scheme, denote $X^{(p^n)}$ the n-th iterated pullback of $X \to S$ along $F : S \to S$ the absolute Frobenius. Similarly the absolute Frobenius on X induces a relative Frobenius $F^n_{X/S} : X \to X^{(p^n)}$ and is canonical equal to

$$F^n_{X/S} = F_{X^{(p^{n-1})}/S} \circ \ldots \circ F_{X/S}. \tag{2.1}$$

As we know from the last talk, the Cartier inverse becomes an isomorphism if $X \to S$ is smooth. In this case we set the sheaf of abelian groups as follows.

Definition 2.1 Let

$$\begin{aligned} B_0\Omega^i_{X/S} := 0, B_1\Omega^i_{X/S} := d\Omega^{i-1}_{X/S} \\ Z_0\Omega^i_{X/S} := \Omega^i_{X/S}, Z_1\Omega^i_{X/S} := \Omega^i_{X/S,d=0} \end{aligned} \tag{2.2}$$

and iteratively

$$\begin{aligned} B_n\Omega^i_{X^{(p)}/S} \underset{\simeq}{\overset{C^{-1}_{X/S}}{\to}} B_{n+1}\Omega^i_{X/S}/B_1\Omega^i_{X/S} \\ Z_n\Omega^i_{X^{(p)}/S} \underset{\simeq}{\overset{C^{-1}_{X/S}}{\to}} Z_{n+1}\Omega^i_{X/S}/B_1\Omega^i_{X/S}. \end{aligned} \tag{2.3}$$

Inspecting the concrete behaviour of $C^{-1}_{X/S}$, we see that $Z_n\Omega^i_{X/S}$ and $B_n\Omega^i_{X/S}$ have a structure of $\mathcal{O}_{X^{(p^n)}}$-modules and the Cartier inverse induces an isomorphism of $\mathcal{O}_{X^{(p^{n+1})}}$-modules:

$$C_{X/S}^{-1} : Z_n\Omega^i_{X^{(p)}/S}/B_n\Omega^i_{X^{(p)}/S} \xrightarrow{\simeq} Z_{n+1}\Omega^i_{X/S}/B_{n+1}\Omega^i_{X/S}. \tag{2.4}$$

For $\pi : X \to S$ the n-th Cartier inverse is a map

$$C_{X/S}^{-n} : \Omega^i_{X^{(p^n)}/S} \to Z_n\Omega^i_{X/S} \tag{2.5}$$

and if π if smooth then it induces an isomorphism

$$C_{X/S}^{-n} : \Omega^i_{X^{(p^n)}/S} \xrightarrow{\simeq} Z_n\Omega^i_{X/S}/B_n\Omega^i_{X/S}. \tag{2.6}$$

We can describe n-th "closed" and "exact" forms in terms of logarithmic differentials $d\log x := dx/x$.

Proposition 2.2

1. *$Z_n\Omega^i_{X/S}$ and $B_n\Omega^i_{X/S}$ are all free coherent $\mathcal{O}_{X^{(p^n)}}$-module compatible with base change.*
2. *$B_n\Omega^i_{X/S}$ is locally generated by sections $x_1^{p^r-1}...x_i^{p^r-1}dx_1...dx_i$ with $x_1, ..., x_i \in \mathcal{O}_X$ and $0 \leq r \leq n-1$. Equivalently, is generated by $x_1^{p^r} d\log x_1...d\log x_i$ for $x_1, ..., x_i \in \mathcal{O}_X^\times$ and $0 \leq r \leq n-1$.*
3. *$Z_n\Omega^i_{X/S}$ is locally generated by $B_n\Omega^i_{X/S}$ and sections $ax_1^{p^n-1}...x_i^{p^n-1}dx_1...dx_i$ for $x_1, ..., x_i \in \mathcal{O}_X$ and $a \in \mathcal{O}_{X^{(p^n)}}$, or equivalently $a d\log x_1...d\log x_i$ for $x_1, ..., x_i \in \mathcal{O}_X^\times$ and $a \in \mathcal{O}_{X^{(p^n)}}$.*

Proof. We prove the first statement by induction, $n = 0, 1$ is trivial or handled by Cartier. As these sheaves are étale sheaves, we reduce to the case where $X = \mathbb{A}^n_S$. By base change we can further reduce to $\mathbb{A}^n_{\mathbb{F}_p}$. Fix an algebraic closure k of $\mathbb{F}_p$ and denote $\overline{X}$ be the extension of scalars. Then $Z_n\Omega^i_{X/S}$ and $B_n\Omega^i_{X/S}$ are coherent in one open dense subset by [[Sta26], 051S]. Translating by k-points we show they are free everywhere.

For the second statement. Still there is nothing to prove for $n \leq 1$. We calculate

$$\begin{aligned} C_{X/S}^{-1}\left(x_1^{p^r-1}...x_i^{p^r-1}dx_1...dx_i\right) &= x_1^{p^{r-1}-1}...x_i^{p^{r-1}-1}dx_1...dx_i \bmod B_1 \\ C^{-1}\left(x^{p^r} d\log x_1...d\log x_i\right) &= x_1^{p^{r-1}} d\log x_1...d\log x_i \bmod B_1 \end{aligned} \tag{2.7}$$

together with inductions we shows 2.

Finally 3. is analog to 2. ■

3 Étale-local description of $v(r)$

This section was not covered in [Mil76], but Milne gave some evidences of the description of $v(r)$ using logarithmic differentials (he only computed for $r = 2$). This was later worked out by Bloch and was written in [Ill79], thus we can justify the name of $v(r)$ and extend to higher torsion groups.

The following theorem is due to Bloch.

Theorem 3.1 *[[Ill79], 0.2.4.2] For $r \geq 1$, the sheaf $v(r)$ is an étale sheaf over X and is (étale locally) generated by logarithmic differentials, i.e. sections of the form $d\log x_1 ... d\log x_r$ for $x_1, ..., x_r \in \mathcal{O}_X^\times$.*

Proof. Wlog we can assume S is affine and write it as a filtered colimit of $\mathbb{F}_p$-algebras of finite type. Since $W^* - C_{X/S}$ commutes with filtered colimits, it suffices to show the theorem for S an $\mathbb{F}_p$-algebra of finite type and X a scheme of finite type over S.

Next we reduce to the case where S is reduced.

Lemma 3.2 *Let $I \subset S$ a square zero ideal and S' the associated closed subscheme of S. Suppose the theorem holds for $X_{S'}/S'$, then it holds for X/S.*

Proof. Let ω be a closed r-form of X over S and ω' be the image of ω in $\Omega^r_{X_{S'}/S}$. Suppose $\omega' = \sum d\log x_i$, Since $\mathcal{O}_X^\times \to \mathcal{O}_{X_{S'}}^\times$ is surjective and $U' \to X' \to X$ is étale for each étale map $U \to X'$, by linearity we see (after localize in étale topology) it suffices to assume $\omega' = 0$. Then $\omega \in I\Omega^r_{X/S}$ and $\omega^p = 0$, thus ω is killed by W^*. By definition we have

$$v(r) = \ker\left(C_{X/S}\right) \cap I\Omega^r_{X/S,d=0} = I\Omega^r_{X/S,\text{ ex}} \tag{3.1}$$

by the definition of Cartier operator.

Now according to Proposition 2.2 every section of exact forms is locally a sum of the form $dx_1 d\log x_2 ... d\log x_r$. It suffices to show for $a \in I, x \in \mathcal{O}_X$, adx is a logarithmic differential. Notice as $a^2 = 0$ we have

$$adx = -d\log(1 - ax) \tag{3.2}$$

and we can conclude. ■

Now we assume S is reduced, then W^* is injective. Then the exact sequence

$$0 \to v(r) \to \Omega^r_{X/S,d=0} \overset{1-C}{\to} \Omega^r_{X/S} \tag{3.3}$$

yields an exact sequence

$$0 \to v(r) \to \Omega^r_{X/S,d=0}/\Omega^r_{X/S,\text{ ex}} \overset{W^*-C}{\to} \Omega^r_{X^{(p)}/S}/\Omega^r_{X^{(p)}/S,\text{ ex}}. \tag{3.4}$$

Let L be the free sheaf of $\mathbb{F}_p$-algebra generated by $\mathcal{O}_X^\times$. For a section $x \in \mathcal{O}_X^\times$, we denote $\underline{d\log x}$ be the image in L. The $\mathbb{F}_p$-linear map

$$\prod_{1 \leq i \leq r} \mathcal{O}_X^\times \to \Omega^r_{X/S,d=0}, (x_1, ..., x_i) \mapsto d\log x_1 ... d\log x_i \tag{3.5}$$

induces an $\mathbb{F}_p$-linear map $\Lambda^r L \to \Omega^r_{X/S,d=0}$ and by base change an $\mathcal{O}_{X^{(p)}}$-linear map

$$\varphi : \mathcal{O}_{X^{(p)}} \otimes \Lambda^r L \to \mathcal{H}^r\left(\Omega^{\cdot}_{X/S}\right) \tag{3.6}$$

that sends $\left(a\underline{d\log} x_1 ... \underline{d\log} x_r\right)$ to $[a d\log x_1 ... d\log x_r]$.

Lemma 3.3 *φ is surjective.*

Proof. Since $\mathcal{O}_X$ is additively generated by $\mathcal{O}_X^{\times}$, every section of $\Omega^r_{X^{(p)}/S}$ is a sum of $aW^*(d\log x_1 ... d\log x_r)$ with $a \in \mathcal{O}_{X^{(p)}}$ and $x_1, ..., x_r \in \mathcal{O}_X^{\times}$. Apply $C^{-1}_{X/S}$ we see it sends such a section to $[a d\log x_1 ... d\log x_r]$ and we may conclude since $C^{-1}_{X/S}$ is an isomorphism. ■

On the other hand, pulling back along W^* gives an $\mathcal{O}_{X^{(p)}}$-linear map

$$u : \mathcal{O}_{X^{(p)}} \otimes \Lambda^r L \to \Omega^r_{X^{(p)}/S} \tag{3.7}$$

and induces an $\mathcal{O}_{X^{(p)}}$-linear map

$$\psi : \mathcal{O}_{X^{(p)}} \otimes \Lambda^r L \to \Omega^r_{X^{(p)}/S} / d\Omega^{r-1}_{X^{(p)}/S}. \tag{3.8}$$

These two maps are also surjective.

From the commutative diagram

$$\begin{array}{ccc}
\mathcal{O}_{X^{(p)}} \otimes \Lambda^r L & \xrightarrow{\varphi} & \mathcal{H}^r\left(\Omega^{\cdot}_{X/S}\right) \\
{\scriptstyle (1-F)\otimes 1}\downarrow & & \downarrow{\scriptstyle W^* - C_{X/S}} \\
\mathcal{O}_{X^{(p)}} \otimes \Lambda^r L & \xrightarrow{\psi} & \Omega^r_{X^{(p)}/S} / d\Omega^{r-1}_{X^{(p)}/S}
\end{array}$$

we have a commutative diagram with exact columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R^r & \longrightarrow & Q^r & \longrightarrow & v(r) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N^r & \longrightarrow & \mathcal{O}_{X^{(p)}} \otimes \Lambda^r L & \xrightarrow{\varphi} & \mathcal{H}^r\left(\Omega^{\cdot}_{X/S}\right) & \longrightarrow & 0 \\
 & & \downarrow & & {\scriptstyle (1-F)\otimes 1}\downarrow & & \downarrow{\scriptstyle W^* - C_{X/S}} & & \\
0 & \longrightarrow & P^r & \longrightarrow & \mathcal{O}_{X^{(p)}} \otimes \Lambda^r L & \xrightarrow{\psi} & \Omega^r_{X^{(p)}/S} / d\Omega^{r-1}_{X^{(p)}/S} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Essentially we need to show $Q^r \to v(r)$ is surjective, which is equivalent of showing $N^r \to P^r$ is surjective. For $r \leq 1$ this is again clear by Lemma 3.3. We need to prove the following lemma.

Lemma 3.4 *For $r \geq 2$, P^r is generated by $y\underline{d\log}x_2 ... \underline{d\log}x_r$ for $x_2, ..., x_r \in \mathcal{O}_X^\times$ and $y \in P^1$.*

Proof. Let $M^r := \ker(u) \subset P^r$ and $z \in P^r$, then $u(z) \in d\Omega^{r-1}_{X^{(p)}/S}$. From Proposition 2.2 we see it is a sum of $\lambda W^*(dx_1 d\log x_2 ... d\log x_r)$ for $\lambda \in \mathcal{O}_S$ and $x_1, ..., x_r \in \mathcal{O}_X^\times$. Since $\lambda W^*(x_1)\underline{d\log}x_1 \in P^1$, wlog we may assume $u(z) = 0$, hence $z \in M^r$. But exterior product is right exact, hence M^r is determined by M^1, hence z is generated by $y\underline{d\log}x_2 ... \underline{d\log}x_r$ with $y \in M^1$ and $x_2, ..., x_r \in \mathcal{O}_X^\times$. ∎

In one remark by Illusie, he claimed he did not know whether $v(r)$ is Zariski-locally generated by logarithmic differentials. Nowadays we known this is true by Bloch-Gabber-Kato as in [[BK86], Theorem 2.1], where the correct intermediate used was the Milnor K-theory $\bmod p$.

The proof of this fact is much more elegant. The injectivity follows by descent to a purely transcendental extension of a perfect field. And the surjectivity follows by reduce to a semi-localized Dedekind domain (so passing to radical we have a well defined base) and do the induction. In particular, this isomorphism still holds for higher p-torsion, which we will handle in the next talk.

Bibliography

[Ber81] P. Berthelot, "Le Théorème de Dualité Plate Pour les Surfaces (d'après J.S. Milne)," *Surfaces Algébriques*, vol. 868. Springer Berlin Heidelberg, Berlin, Heidelberg, pp. 203–237, 1981. doi: 10.1007/BFb0090649.

[BK86] S. Bloch and K. Kato, "p-adic Étale cohomology," *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, vol. 63, no. 1, pp. 107–152, Dec. 1986, doi: 10.1007/BF02831624.

[DGB70] M. Demazure, P. Gabriel, and J. E. Bertin, *Propriétés générales des schémas en groupes*, vol. 1. in Schemas En Groupes. Seminaire de Geometrie Algebrique Du Bois Marie 1962/64 (SGA 3), vol. 1. Berlin, Heidelberg: Springer, 1970. doi: 10.1007/BFb0058993.

[Ill79] L. Illusie, "Complexe de de Rham-Witt et cohomologie cristalline," *Annales scientifiques de l'É.N.S.*, vol. 12, no. 4, 1979.

[Mil76] J. S. Milne, "Duality in the flat cohomology of a surface," *Annales scientifiques de l'École normale supérieure*, vol. 9, no. 2, pp. 171–201, 1976, doi: 10.24033/asens.1309.

[Sta26] Stacks project authors, "The Stacks project." [Online]. Available: https://stacks.math.columbia.edu/